

Analytic Gröbner fan and its applications

Rouchdi Bahloul (Lyon) and Nobuki Takayama (高山, Kobe)



Notions in polyhedral geometry (1)

References: G.Ziegler, Lectures on polytopes. R.Thomas, Lectures in geometric combinatorics.

1. **Polyhedron** is the intersection of finite closed half spaces in \mathbf{R}^d .
2. **Polytope** is a bounded polyhedron.
3. Closed **cone** is the intersection of finite closed half spaces of the form $\sum_{j=1}^d c_{ij}x_j \geq 0$.
4. Closed cone C can be expressed as

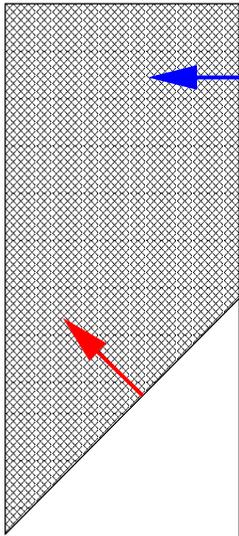
$$C = \mathbf{R}_{\geq 0} a_1 + \cdots + \mathbf{R}_{\geq 0} a_n$$

where $a_i \in \mathbf{R}^d$ and $\mathbf{R}_{\geq 0} = \{x \in \mathbf{R} \mid x \geq 0\}$.

Notions in polyhedral geometry (1) in picture

Closed **cone** is the intersection of finite closed half spaces of the form $\sum_{j=1}^d c_{ij}x_j \geq 0$.

The intersection of $-x \geq 0$ and $-x + y \geq 0$ is



and can be expressed as

$$\mathbf{R}_{\geq 0} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathbf{R}_{\geq 0} \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Notions in polyhedral geometry (2)

1. Let P be a polyhedron and $w \in \mathbf{R}^d$ a weight vector.

$$\begin{aligned} \text{face}_w(P) &:= \{m \in P \mid \langle m, w \rangle \leq \langle x, w \rangle \text{ for all } x \text{ in } P\} \\ &= \text{The lowest points of } P \text{ with respect to } w. \end{aligned}$$

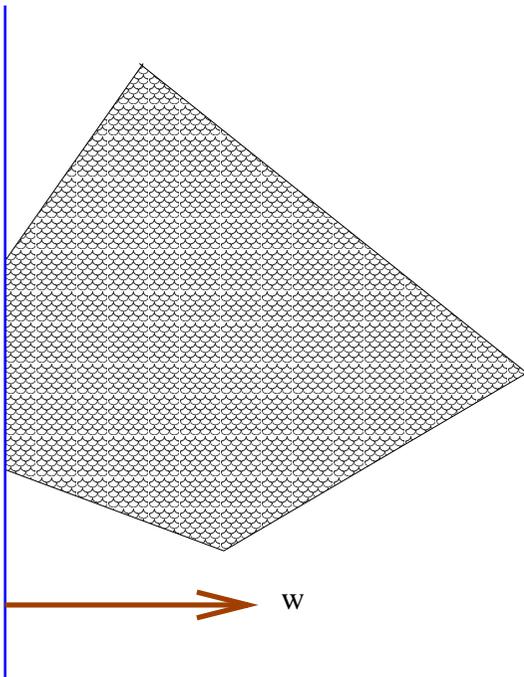
2. $\text{face}_w(P)$ is called the **face** of P determined by w .

3. Although choices for w are infinite, there are only finite faces.

4. The face is an analogous notion of the initial term for a polynomial f . In fact, the Newton polygon of $\text{in}_w(f)$ agrees with the $\text{face}_w(\text{New}(f))$.

Notions in polyhedral geometry (2) in picture

$\text{face}_w(P) := \{m \in P \mid \langle m, w \rangle \leq \langle x, w \rangle \text{ for all } x \text{ in } P\}$
= The lowest points of P with respect to w .



In this example, $w = (1, 0)$.

For $f = \sum_{\alpha \in E} c_{\alpha} x^{\alpha}$, $c_{\alpha} \neq 0$, $\alpha \in E$, $x^{\alpha} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $w \in \mathbf{R}^d$ we define w -order

$$\text{ord}_w(f) = \min_{\alpha \in E} \langle w, \alpha \rangle$$

and the **initial form** of f with respect to w

$$\begin{aligned} \text{in}_w(f) &:= \sum_{\alpha \in E, \langle \alpha, w \rangle = \text{ord}_w(f)} c_{\alpha} x^{\alpha} \\ &= \text{The lowest order terms of } f \text{ with respect to } w. \end{aligned}$$

Example: $\text{in}_{(2,1)}(x_1 x_2 + 3x_2^3 + x_1^2) = x_1 x_2 + 3x_2^3$.

The **initial ideal** $\text{in}_w(I)$ is the ideal generated by initial forms $\text{in}_w(f)$, $f \in I$.

We may use the following alternative definition

$$\begin{aligned} \text{face}_w(P) &:= \{m \in P \mid \langle m, w \rangle \geq \langle x, w \rangle \text{ for all } x \text{ in } P\} \\ &= \text{The highest points of } P \text{ with respect to } w. \end{aligned}$$

In this case, we change the definition of the initial form as

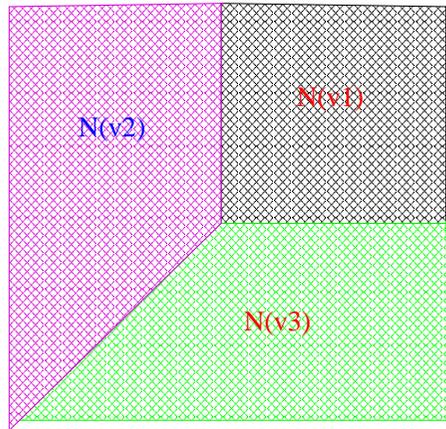
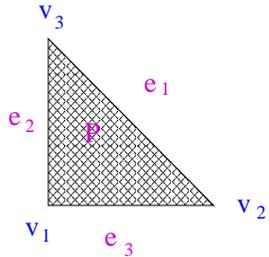
$$\text{in}_w(f) = \text{The highest order terms of } f \text{ with respect to } w.$$

Notions in polyhedral geometry (3)

1. Let e be a face of P . $N(e) = \{w \in \mathbf{R}^d \mid \text{face}_w(P) = e\}$ (**normal cone**).
2. Let F be the set of all faces of P . The collection of $\bar{N}(e)$, $e \in F$ is called the **normal fan** of P and denoted by $\bar{\mathcal{N}}(P)$.
3. A finite set Δ of closed cones are called a (polyhedral) **fan** when
 - (a) If $\sigma \in \Delta$, then any face of σ is in Δ .
 - (b) If $\sigma \in \Delta$ and $\tau \in \Delta$, then $\sigma \cap \tau$ is a face of both of σ and τ .
4. **Theorem.** The normal fan is a fan.

Notions in polyhedral geometry (3) in picture

Let e be a face of P . $N(e) = \{w \in \mathbf{R}^d \mid \text{face}_w(P) = e\}$.



$$N(v_1) = \mathbf{R}_{>0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{R}_{>0} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$N(v_2) = \mathbf{R}_{>0} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathbf{R}_{>0} \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$N(v_3) = \mathbf{R}_{>0} \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \mathbf{R}_{>0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$N(e_1) = \mathbf{R}_{>0} \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

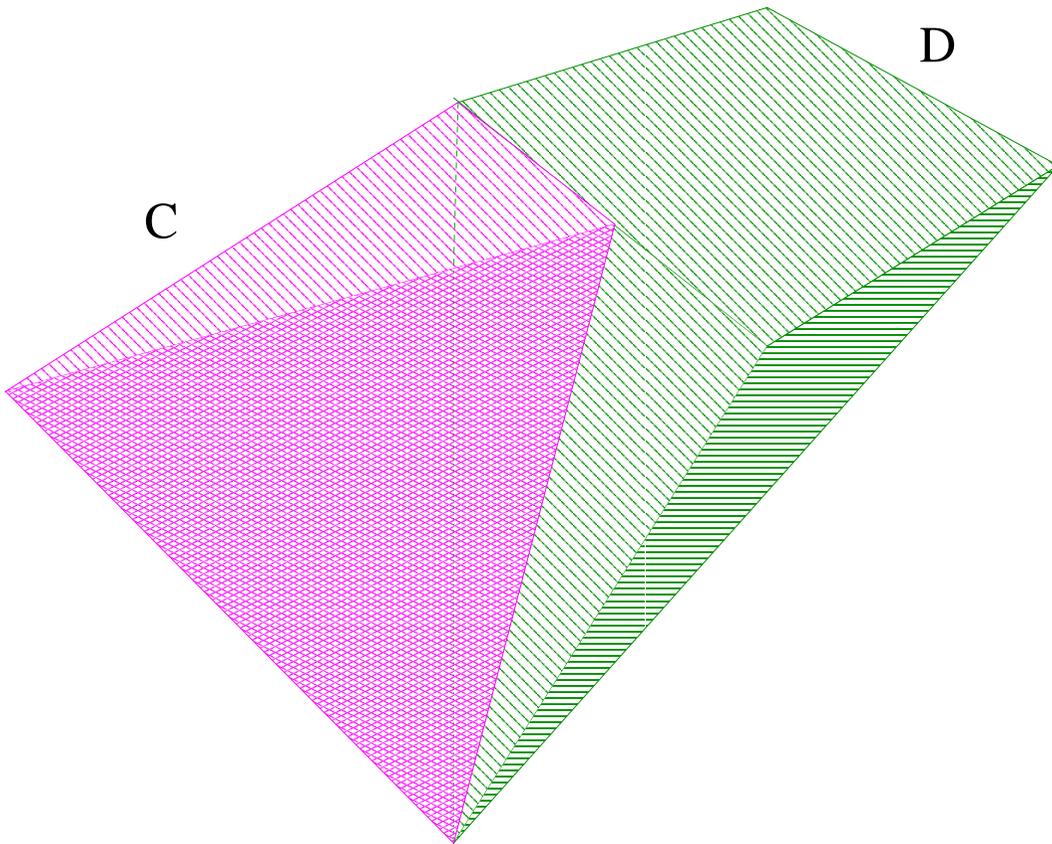
$$N(e_2) = \mathbf{R}_{>0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$N(e_3) = \mathbf{R}_{>0} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad N(P) = (0, 0)$$

What is a fan? What is not a fan?

A finite set Δ of closed cones are called a **fan** when

1. If $\sigma \in \Delta$, then any face of σ is in Δ .
2. If $\sigma \in \Delta$ and $\tau \in \Delta$, then $\sigma \cap \tau$ is a face of both of σ and τ .



$$\Delta = \{C, D, \text{faces of } C \text{ and } D\}$$

is not a fan. Because, $C \cap D$ is not a face of D .

A brief history on Gröbner fan (1).

Let I be a **homogeneous ideal** in $\mathbf{K}[x_1, \dots, x_d]$ and w a weight vector. Let $e = \text{in}_w(I)$ be the initial ideal of I with respect to w .

$$C[e] = C[w] := \{w \in \mathbf{R}^d \mid \text{in}_w(I) = e\}$$

is called the open (global) **Gröbner cone** for e (for w). **the normal cone** $N(e) = \{w \in \mathbf{R}^d \mid \text{face}_w(P) = e\}$

1. There exist only finite distinct initial ideals. **Finite faces.**
2. $\bar{C}[e]$ is a rational polyhedral cone. (T.Mora, L.Robbiano, 1988)
3. " $\bar{\mathcal{E}}(I) :=$ the collection of closed Gröbner cones" is called a **Gröbner fan**.
4. The Gröbner fan is a fan (B.Sturmfels, Gröbner basis and convex polytopes, 1995). **The normal fan is a fan.**

Example: The Gröbner fan of $\langle f \rangle$ is the normal fan of the Newton polygon of f . $f = 1 + x_1 + x_2$

Theorem (B.Sturmfels, 1990): The Gröbner fan of homogeneous affine toric ideal is a refinement of the secondary fan standing for the regular triangulations.

A brief history on Gröbner fan (2).

1. K.Fukuda, A.Jensen, R.Thomas, Computing Gröbner fans, [math.AC/0509544](#).
A definition of (global) Gröbner fan for **inhomogeneous** ideal.
Theorem: The Gröbner fan is a fan.
2. Collection of $\bar{C}[\text{in}_w(I)]$ such that $\text{in}_w(I)$ does not contain a monomial is called the **tropical zero set** associated to I , e.g., T.Bogart, A.Jensen, D.Speyer, B.Sturmfels, R.Thomas, Computing tropical varieties, [math.AG/0507563](#).
Theorem: The tropical zero set can be regarded as a subfan of the Gröbner fan.

“global” : Ideals are in the polynomial ring $K[x_1, \dots, x_d]$ or in the Weyl algebra. The initial forms are also in $K[x_1, \dots, x_d]$.

$$\begin{aligned} \hat{\mathcal{O}}_0 &= \mathbf{C}[[x_1, \dots, x_n]], & \mathcal{O}_0^{an} &= \text{germ of convergent power series} \\ \mathcal{O}_0^{alg} &= \{f/g \mid f, g \in \mathbf{C}[x], g(0) \neq 0\} \\ h_{(0,1)}(\hat{\mathcal{D}})_0 &= \hat{\mathcal{O}}_0[h] \langle \partial_1, \dots, \partial_n \rangle \\ &\text{where } \partial_i a(x) = a(x) \partial_i + h \frac{\partial a}{\partial x_i} \end{aligned}$$

$$W_{loc} = \{(w_1, \dots, w_n) \mid w_i \geq 0\}$$

Assi, Castro, Granger (2001) suggested a definition of local Gröbner fan (analytic standard fan) and proved the finiteness and convexity of the local Gröbner cones (analytic standard cones).

Theorem: the set of the local Gröbner cones (= local Gröbner fan) is a polyhedral fan.

Take $w = (0, \dots, 0, w_k, \dots, w_n) \in W_{loc}$, $w_k > 0, \dots,$

$$\text{gr}^w(\hat{\mathcal{O}}) = \mathbf{C}[[x_1, \dots, x_{k-1}]] [x_k, \dots, x_n]$$

Fix an ideal I in $\hat{\mathcal{O}}$. $\text{in}_w(I) \subseteq \text{gr}^w(\hat{\mathcal{O}})$.

$$w \sim w' \Leftrightarrow \text{gr}^w(\hat{\mathcal{O}}) = \text{gr}^{w'}(\hat{\mathcal{O}}) \text{ and } \text{in}_w(I) = \text{in}_{w'}(I)$$

$$C[w] = \{w' \mid w \sim w'\} \quad (\text{local or analytic Gröbner cone})$$

$$\bar{\mathcal{E}}(I) = \{\bar{C}[w]\} \quad (\text{local or analytic Gröbner fan})$$

Q. Why do we need to prove that $\bar{\mathcal{E}}$ is a polyhedral fan?

(1) It is a natural property. It should be proved.

(2) Since $\bar{\mathcal{E}}$ is a polyhedral fan, we may only enumerate the maximal dimensional cones. \Rightarrow Effective enumeration.

(3) Corollary. Let L be a linear subspace of \mathbf{R}^n . Then, $\bar{\mathcal{E}}(I) \cap L$ is a polyhedral fan.

Comparison (1)

1. For homogeneous I , local and global cones agree.
2. For polynomial generators, local and analytic agree.

Theorem: Local (analytic) Gröbner fan is computable if the ideal is generated by polynomials.

1. (Double) homogenization.
2. Tangent cone algorithms in \mathcal{O}^{alg} (Mora, Gräbe, Greuel, Pfister) and \mathcal{D}^{alg} (Grangner-Oaku-T).

Example. Global vs. local

$$\begin{aligned} I^{global} &= K[x_1, x_2, x_3](x_1 + x_2) + K[x_1, x_2, x_3](1 - x_3) \\ I^{an} &= \hat{\mathcal{O}} \cdot (x_1 + x_2) + \hat{\mathcal{O}} \cdot (1 - x_3) = \langle 1 \rangle \end{aligned}$$

$$\begin{aligned} \text{in}_{(2,1,0)}(x_1 + x_2) &= x_2 \\ \text{in}_{(2,1,0)}(1 - x_3) &= 1 - x_3 \end{aligned}$$

$$\begin{aligned} \text{in}_{(2,1,0)}(I^{global}) &= K[x_1, x_2, x_3]x_2 + K[x_1, x_2, x_3](1 - x_3) \\ \text{in}_{(2,1,0)}(I^{an}) &= K[x_1, x_2][[x_3]] \cdot 1 \end{aligned}$$

$$\begin{aligned} \text{in}_{(1,2,0)}(I^{global}) &= K[x_1, x_2, x_3]x_1 + K[x_1, x_2, x_3](1 - x_3) \\ \text{in}_{(1,2,0)}(I^{an}) &= K[x_1, x_2][[x_3]] \cdot 1 \end{aligned}$$

Q. Why do we need $h_{(0,1)}(\widehat{\mathcal{D}}) = \widehat{\mathcal{O}}[h]\langle \partial_1, \dots, \partial_n \rangle$?

Informally, it is like that we need a homogeneous input in $\mathbb{C}[[x_1, \dots, x_n]][x_{n+1}, \dots, x_{2n}]$ to get the reduced Gröbner basis for local orders.

There will be no polyhedral Gröbner fan for \mathcal{D} , because the division theorem does not hold.

Example (T.Oaku):

$n = 1$, $u + v \geq 0$, $u < 0$. u is the weight for x and v is the weight for ∂ .

Put $f = x$ and $g = x + x^k \partial$ where k is taken so that $x > x^k \partial$.

Suppose that we can perform a division of the form

$$f = qg + r = q(x + x^k \partial) + r(\partial)$$

where the remainder r does not contain the variable x since the leading term of g is x . Take the initial of $f - qg$ with the weight vector $(0, 1)$, then it is of the form $x^k \xi$. It is a contradiction.

References:

1. R.Bahloul and N.Takayama, Local Gröbner fan: polyhedral and computational approach, `math.AG/0412044`
2. R.Bahloul and N.Takayama, Local Gröbner fan: to appear in C.R. Academy Sci. Paris.
<http://www.math.kobe-u.ac.jp/~taka>

Application 1: A new algorithm to get local Bernstein-Sato polynomials (by R.Bahloul. computational details are by H.Nakayama(中山) and T)

Rouchdi Bahloul gives a constructive proof of the existence of local Bernstein-Sato polynomial in terms of the local Gröbner fan. (R. Bahloul, *Démonstration constructive de l'existence de polynômes de Bernstein-Sato pour plusieurs fonctions analytiques*, Compositio Math. **141** (2005), 175–191.) Our construction algorithm of local Gröbner fan gives a method to compute local Bernstein-Sato polynomials in two variable case.

Switch “max” and “min” ($w \Leftrightarrow -w$) in the sequel.

Given $f_1, f_2 \in \mathbf{C}[x_1, x_2]$ and $v \in \mathbf{N}_0^2$. v is usually $(1, 1)$ or $(1, 0)$ or $(0, 1)$.

Polynomials b satisfying

$$b(s_1, s_2) f_1^{s_1} f_2^{s_2} \in \mathcal{D}[s] f_1^{s_1+v_1} f_2^{s_2+v_2}$$

are called Bernstein-Sato polynomials for f_1, f_2, v . The set of all BS polynomials is an ideal and is denoted by $\mathcal{B}^v(f_1, f_2)$.

Problem: Compute non-zero $b(s)$.

$$I = \langle t_1 - f_1, t_2 - f_2, \partial_{x_i} + \frac{\partial f_1}{\partial x_i} \partial_{t_1} + \frac{\partial f_2}{\partial x_i} \partial_{t_2}, i = 1, 2 \rangle$$

$$\bar{B}(I) = \bar{\mathcal{E}}(I) \cap \{ (\overbrace{-w_1}^{t_1}, \overbrace{-w_2}^{t_2}, \overbrace{0}^{x_1}, \overbrace{0}^{x_2}, \overbrace{w_1}^{\partial_{t_1}}, \overbrace{w_2}^{\partial_{t_2}}, \overbrace{0}^{\partial_{x_1}}, \overbrace{0}^{\partial_{x_2}}) \mid w_i \in \mathbf{R}_{\geq 0} \}$$

We call $\bar{B}(I)$, the **local BSS Gröbner fan**.

Theorem-Algorithm

$$\prod_{L \in \text{ray in } \bar{B}(I), -\langle L, v + \kappa \rangle < k \leq 0} b_L(L_1 s_1 + L_2 s_2 - k)$$

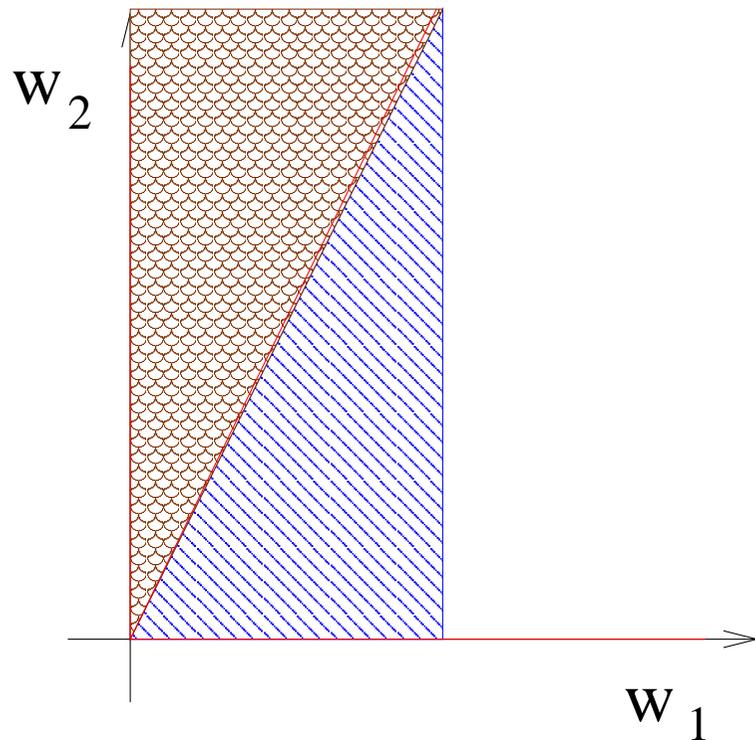
lies in $\mathcal{B}^v(f_1, f_2)$. Here κ is a vector computed from the local BSS fan and b_L is a local b -function along L .

Example: $f_1 = x_1 + x_2$, $f_2 = x_1^2 + x_2^2$. $\bar{B}(I)$ consists of two maximal dimensional cones $(1, 0), (1, 2)$ and $(1, 2), (0, 1)$.

For $L = (1, 2)$, $b_L(s)$ is $2s^2 + s$.

For $L = (1, 0)$, $b_L(s)$ is s .

For $L = (0, 1)$, $b_L(s)$ is s^2 .



Application 2: local tropical variety. N.Touda(富田), Local Tropical Variety, math.AG/0511486

$\text{Trop}(I) = \{w \in \bar{\mathcal{E}}(I) \mid \text{in}_w(I) \in \text{gr}^w(\hat{\mathcal{O}}) \text{ does not contain a monomial}\}$

$\text{Trop}(I)$ is called the local tropical zero set.

Theorem (N.Touda): $\text{Trop}(I)$ can be regarded as a sub fan of the local Gröbner fan.

Algorithm(conjectural). $\text{Trop}(I)$ can be computed if I is generated by polynomials.

Open problems. Natural geometric definition. ...

Application 3: Slopes

$$V_1 = (\overbrace{-1}^{x_1}, \overbrace{0}^{x_2}, \dots, \overbrace{0}^{x_n}; \overbrace{1}^{\partial x_1}, \overbrace{0}^{\partial x_2}, \dots, \overbrace{0}^{\partial x_n}), F = (\overbrace{0}^{x_1}, \overbrace{0}^{x_2}, \dots, \overbrace{0}^{x_n}; \overbrace{1}^{\partial x_1}, \overbrace{1}^{\partial x_2}, \dots, \overbrace{1}^{\partial x_n})$$

$$\mathcal{I}_1(I) = \bar{\mathcal{E}}(I) \cap \{pF + qV_1 \mid p, q \geq 0\}$$

We call $\mathcal{I}_1(I)$ the **irregularity local Gröbner fan** along $x_1 = 0$.

The irregularity global Gröbner fan is a refinement of the irregularity local Gröbner fan. The irregularity local Gröbner fan is a refinement of the fan by the slopes.

Slopes are introduced by Y.Laurent and Z.Mebkhout. Constructive method to compute slopes is given by Assi-Castro-Granger (How to compute slopes). U.Walther's talk.

Application 4: Analytic GKZ compactification and series solutions

Theorem ([SST; Chapter2]) Global small Gröbner fan of regular holonomic I gives a natural compactification to construct series solutions of I .

$\bar{\mathcal{E}}(I) \cap \{(-w, w) \mid w \in \mathbf{R}_{\geq 0}\}$ is called the **analytic small Gröbner fan**.

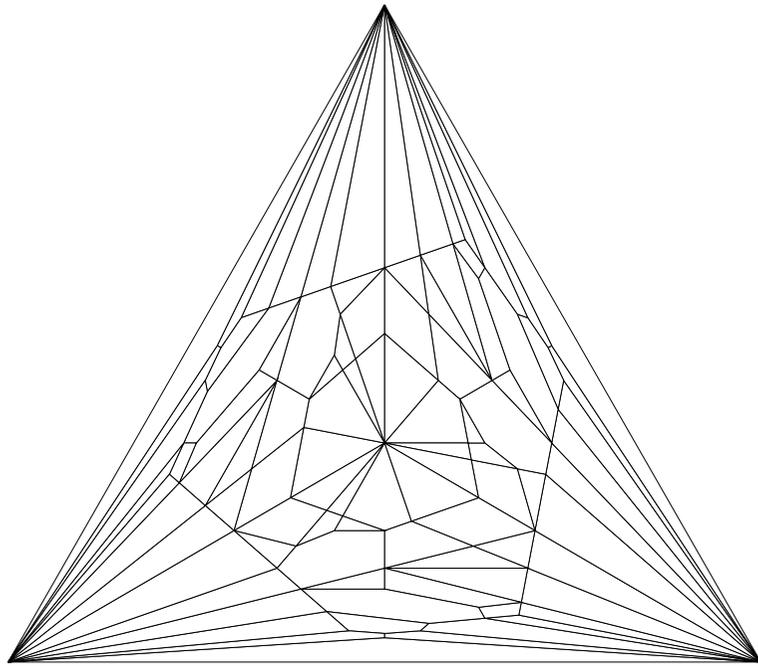
Theorem (conjectural) Analytic small Gröbner fan of regular holonomic I gives a natural compactification to construct series solutions of I .

[SST] Saito-Sturmfels-T, Gröbner deformations of hypergeometric differential equations.

Q. Implementation?

Computation is split into a system for algebra ([Kan/sm1](#)) and a system for geometry ([Polymake](#)). They are connected by the [OpenXM-RFC 104 protocol](#) (OoHG, OpenXM on HTTP/GET). This design gives us [a robust system](#), since the polymake is a strong, flexible, and robust system for polytopes developed by E. Gawrilow and M. Joswig linked with GMP and cdd by K. Fukuda. We use polymake properties DIM, FACETS, INEQUALITIES.

1. Fastest: gfan by Anders Jensen.
2. Broadest scope: gfan.sm1 on kan/sm1



Gröbner fan of $\langle a^2b - c - a, b^2c - a - b, c^2a - b - c \rangle$ computed by gfan

Input example.

```
% BS poly.  $x+y$ ,  $x^2+y^2$ 
/cone.loaded boundp { }
{
  [(parse) (cohom.sm1) pushfile] extension
  [(parse) (dhecart.sm1) pushfile] extension
  [(parse) (gfan.sm1) pushfile] extension
  /cone.loaded 1 def
} ifelse

%-----Globals-----
% 0 :  $x,y,Dx,Dy$ 
% 1 :  $x,y,Dx,Dy,h,H$ 
% 2 :  $x,y,Dx,Dy,h$ 
/cone.type 2 def

/cone.local 1 def

% cone.h0: 1 ==> weight(h)=0
/cone.h0 1 def

% cone.input : input polynomials
/cone.input
[
  (t1-x-y) (t2-x^2-y^2) (2*x*Dt2+Dt1+Dx) (2*y*Dt2+Dt1+Dy)
]
```

```

def

/cone.vlist [(t1) (t2) (x) (y) (Dt1) (Dt2) (Dx) (Dy) (h)] def
/cone.vv (t1,t2,x,y) def

/cone.parametrizeWeightSpace {
  4 2 parametrizeSmallFan
} def

% cone.w_start : starting weight
/cone.w_start
  [1 3]
def
%homogenize
  [cone.vv ring_of_differential_operators
  [[(t1) -1 (t2) -1 (Dt1) 1 (Dt2) 1]] ecart.weight_vector
  0] define_ring
  dh.begin
  cone.input { . homogenize toString } map /cone.input set
  dh.end

/cone.DhH 1 def
/cone.gb {
  cone.gb_DhH
} def
% ----- end of configure

```

```
getGrobnerFan  
printGrobnerFan
```

```
dhCones_h          % dehomogenize the fan  
dhcone.rtable  
dhcone.printGrobnerFan
```

The number of cones = 2

```
0 : begin dhcone -----
facets=
  [
    [ -2 , 1 ] % normal vectors for 0th Grobner cone.
    [ 1 , 0 ]
  ]
nextcid=
  [ 1 , -2 ]
cones=
  [ 0 ]
----- end dhcone

1 : begin dhcone -----
facets=
  [
    [ 0 , 1 ] % normal vectors for 1th Grobner cone.
    [ 2 , -1 ]
  ]
nextcid=
  [ -2 , 0 ]
cones=
  [ 1 ]
----- end dhcone

0 : begin gbasis -----
initial=
  [
```

```

    [   -x-y , 2*y*Dt2 , -2*y^2 , 2*t1*Dt2 , t1*Dt1+2*t2*Dt2-y*Dx+y*Dy+2*h , -2*y*Dt1 ,
    [   t1,t2,x,y, , [   t1 , t2 ] , [   ] ]
]
weight=
[   t1 , -1 , t2 , -3 , Dt1 , 1 , Dt2 , 3 ]
----- end gbasis
1 : begin gbasis -----
initial=
[
[   -x-y , 2*Dt1 , -2*y^2 , 4*y*Dt2 , -4*t2*Dt2+2*y*Dx-2*y*Dy-4*h , -4*t1^2*Dt1*Dt2-
[   t1,t2,x,y, , [   t1 , t2 ] , [   ] ]
]
weight=
[   t1 , -12 , t2 , -19 , Dt1 , 12 , Dt2 , 19 ]
----- end gbasis

```