Algorithms for Pfaffian systems and cohomology intersection numbers of hypergeometric integrals*

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Abstract. In the theory of special functions, a particular kind of multidimensional integral appears frequently. It is called the Euler integral. In order to understand the topological nature of the integral, twisted de Rham cohomology theory plays an important role. We propose an algorithm of computing an invariant *cohomology intersection number*. The algorithm is based on the fact that the Euler integral satisfies GKZ system and utilizes algorithms to find rational function solutions of differential equations. We also provide software to perform this algorithm.

Keywords: cohomology intersection numbers · GKZ hypergeometric systems · Gröbner basis.

1 Introduction

In the study of hypergeometric functions in several variables, one often considers the integral of the following form:

$$\langle\omega\rangle = \int_\Gamma h_1(x)^{-\gamma_1}\cdots h_k(x)^{-\gamma_k}x^c\omega, \tag{1}$$

where $h_l(x;z) = h_{l,z^{(l)}}(x) = \sum_{j=1}^{N_l} z_j^{(l)}x^{\mathbf{a}^{(l)}(j)}$ $(l = 1,\dots,k)$ are Laurent polynomials in torus variables $x = (x_1,\dots,x_n)$, $\mathbf{a}^{(l)}(j)\in\mathbb{Z}^n$, $\gamma_l\in\mathbb{C}$ and $c = {}^t(c_1,\dots,c_n)\in\mathbb{C}^n$ are parameters, $x^c = x_1^{c_1}\dots x_n^{c_n}$, Γ is a suitable integration cycle, and ω is an algebraic n-form on $V_z = \{x\in\mathbb{C}^n \mid x_1\dots x_n h_1(x)\dots h_k(x)\neq 0\}$. As a function of the independent variable $z = (z_j^{(l)})_{j,l}$, the integral (1) defines a hypergeometric function. We call the integral (1) the Euler integral representation.

We can naturally define the twisted de Rham cohomology group associated to the Euler-Laplace integral (1). We set $N = N_1+\dots+N_k$, $\mathbb{G}_m^n = \mathrm{Specm}\,\mathbb{C}[x_1^\pm,\dots,x_n^\pm]$, and $\mathbb{A}^N = \mathrm{Specm}\,\mathbb{C}[z_j^{(l)}]$. For any $z\in\mathbb{A}^N$, we can define an integrable connection $\nabla_x = d_x - \sum_{l=1}^k \gamma_l\frac{d_xh_l}{h_l}\wedge + \sum_{i=1}^n c_i\frac{dx_i}{x_i}\wedge : \mathcal{O}_{V_z}\to\Omega^1_{V_z}$. The algebraic de Rham cohomology group $\mathrm{H}^*_{\mathrm{dR}}(V_z;(\mathcal{O}_{V_z},\nabla_x))$ is defined as the hypercohomology group

$$\mathrm{H}^*_{\mathrm{dR}}(V_z;(\mathcal{O}_{V_z},\nabla_x)) = \mathbb{H}^*\left(V_z;(\cdots\overset{\nabla_x}{\to}\Omega^\bullet_{V_z}\overset{\nabla_x}{\to}\cdots)\right). \tag{2}$$

Under a genericity assumption on the parameters γ_l and c, we have the vanishing result $\mathrm{H}^m_{\mathrm{dR}}(V_z;(\mathcal{O}_{V_z},\nabla_x)) = 0$ $(m\neq n)$. Moreover, we can define a perfect pairing $\langle\bullet,\bullet\rangle_{ch} : \mathrm{H}^n_{\mathrm{dR}}(V_z;(\mathcal{O}_{V_z},\nabla_x))\times\mathrm{H}^n_{\mathrm{dR}}\left(V_z;(\mathcal{O}^\vee_{V_z},\nabla^\vee_x)\right)\to\mathbb{C}$ which is called the cohomology intersection form.

The study of intersection numbers of twisted cohomology groups and twisted period relations for hypergeometric functions started with the celebrated work by K. Cho and K. Matsumoto [6]. They clarified that the cohomology intersection number appears naturally as a part of the quadratic relation, a class of functional relations of hypergeometric functions. They also developed a systematic method of computing the cohomology intersection number for 1-dimensional integrals. Since this work, several methods have been proposed to evaluate intersection numbers of twisted cohomology groups, see, e.g., [2], [3], [10], [11], [14], [17], [19] and references therein. All methods utilize comparison theorems of twisted cohomology groups and residue calculus.

We proposed a new method in the paper [16] to obtain cohomology intersection numbers by constructing a rational function solution of a system of linear partial differential equations. One weak point of the method was that it was not algorithmic to construct Pfaffian system (explicit form of integrable connection) for a given basis of the twisted cohomology group. We will give a new algorithm to construct the Pfaffian system for a given basis

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in this paper (Algorithm 1). To our knowledge, algorithms to find the Pfaffian system (or equation) with respect to a given basis of twisted cohomology group do not appear in the literature except the twisted logarithmic cohomology case [1]. Our algorithm works in a general case with a different approach by Saito's b-function [23] and more efficient with a help of polyhedral geometry. The section 2 is a brief overview of the paper [16]. The section 3 is the main part and in the sections 4 and 5, we will give demonstrations of our implementation. As to the construction of rational function solutions, we utilize the algorithm and the implementation by M. Barkatou, T. Cluzeau, C. El Bacha, J.-A. Weil [5] (see also [4],[18] and their references).

2 General results

2.1 The cohomology intersection form

We denote by $\mathrm{H}^n_{dR,c}\left(V_z^{an};(\mathcal{O}_{V_z^{an}},\nabla_x^{an})\right)$ the analytic de Rham cohomology group with compact support. By Poincaré-Verdier duality, the bilinear pairing

$$\begin{array}{ccc} \mathrm{H}^n_{dR,c}\left(V_z^{an};(\mathcal{O}_{V_z^{an}},\nabla_x^{an})\right)\times \mathrm{H}^n_{dR}\left(V_z^{an};(\mathcal{O}^\vee_{V_z^{an}},\nabla_x^{an\vee})\right) & \to & \mathbb{C} \\ \Psi & & \Psi \\ (\phi,\psi) & \mapsto & \int_{V_z^{an}}\phi\wedge\psi \end{array} \tag{3}$$

is perfect. We say the regularization condition is satisfied if the canonical morphism $\mathrm{H}^n_{dR,c}\left(V_z^{an};(\mathcal{O}_{V_z^{an}},\nabla_x^{an})\right)\to \mathrm{H}^n_{dR}\left(V_z^{an};(\mathcal{O}_{V_z^{an}},\nabla_x^{an})\right)$ is an isomorphism. In the following, we always assume that the regularization condition is satisfied. A criterion for this assumption is explained in §2.3. Since $(\mathcal{O}_{V_z},\nabla_x)$ is a regular connection, the canonical morphism $\mathrm{H}^n_{dR}\left(V_z;(\mathcal{O}_{V_z},\nabla_x)\right)\to \mathrm{H}^n_{dR}\left(V_z^{an};(\mathcal{O}_{V_z^{an}},\nabla_x^{an})\right)$ is always an isomorphism by Deligne-Gröthendieck comparison theorem ([7, Corollaire 6.3]). Therefore, we have a canonical isomorphism $\mathrm{reg}:\mathrm{H}^n_{dR}\left(V_z;(\mathcal{O}_{V_z},\nabla_x)\right)\to \mathrm{H}^n_{dR,c}\left(V_z^{an};(\mathcal{O}_{V_z^{an}},\nabla_x^{an})\right)$. Note that the Poincaré dual of the isomorphism reg is called a regularization map in the theory of special functions ([2, §3.2]). Finally, we define the cohomology intersection form $\langle\bullet,\bullet\rangle_{ch}$ between algebraic de Rham cohomology groups by the formula

$$\begin{array}{ccc} \langle\bullet,\bullet\rangle_{ch}:\mathrm{H}^n_{dR}\left(V_z;(\mathcal{O}_{V_z},\nabla_x)\right)\times \mathrm{H}^n_{dR}\left(V_z;(\mathcal{O}^\vee_{V_z},\nabla_x^\vee)\right) & \to & \mathbb{C} \\ \Psi & & \Psi \\ (\phi,\psi) & \mapsto & \int_{V_z^{an}}\mathrm{reg}(\phi)\wedge\psi. \end{array} \tag{4}$$

The value above is called *the cohomology intersection number* of ϕ and ψ.

2.2 The secondary equation

Now, we treat z as a variable. Let $\pi: X=(\mathbb{G}_m)^n_x\times\mathbb{A}^N_z\setminus\bigcup_{l=1}^k\{(x,z)\mid h_{l,z^{(l)}}(x)=0\}\to\mathbb{A}^N_z=Y$ be a natural projection where subscripts stand for coordinates. We define an $\mathcal{O}_Y$-module $\mathcal{H}^n_{dR}$ by the hypercohomology group

$$\mathcal{H}^n_{dR}=\mathbb{H}^n\left(X;(\Omega^\bullet_{X/Y},\nabla_x)\right). \tag{5}$$

Here, $\Omega^\bullet_{X/Y}$ denotes the sheaf of relative differential forms $\oplus_{|I|=\bullet}\mathcal{O}_X dx^I$ with respect to the morphism π. Since Y is affine, $\mathcal{H}^n_{dR}$ is also identified with the sheaf $R^n\pi_*(\Omega^\bullet_{X/Y},\nabla_x)$. For any $z\in U$, there is a natural evaluation morphism $\mathrm{ev}_z:\mathcal{H}^n_{dR}\to\mathrm{H}^n_{dR}\left(V_z;(\mathcal{O}_{V_z},\nabla_x)\right)$. We define the dual object $\mathcal{H}^{n\vee}_{dR}$ by replacing ∇_x by $\nabla_x^\vee$ in the construction above. By the general theory of relative de Rham cohomology group, there exists a non-empty Zariski open subset U of Y such that $\mathcal{H}^n_{dR}\restriction_U\simeq\mathcal{O}_U^{\oplus r}$. Therefore, for any sections $\phi\in\mathcal{H}^n_{dR}\restriction_U$ and $\psi\in\mathcal{H}^{n\vee}_{dR}\restriction_U$, we can define the cohomology intersection number $\langle\phi,\psi\rangle_{ch}$ as a function of $z\in U$ by the formula $U\ni z\mapsto\langle\mathrm{ev}_z(\phi),\mathrm{ev}_z(\psi)\rangle_{ch}\in\mathbb{C}$. This actually defines a $\mathcal{O}_U$-bilinear map $\langle\bullet,\bullet\rangle_{ch}:\mathcal{H}^n_{dR}\restriction_U\times\mathcal{H}^{n\vee}_{dR}\restriction_U\to\mathcal{O}_U$.

We can equip $\mathcal{H}^n_{dR}$ with a structure of a $\mathcal{D}_Y$-module. For this purpose, we only need to define a connection $\nabla^{GM}:\mathcal{H}^n_{dR}\to\Omega^1_Y(\mathcal{H}^n_{dR})$. For any section $\phi\in\mathcal{H}^n_{dR}$, we define

$$\nabla^{GM}\phi=d_z\phi-\sum_{j,l}\gamma_l\frac{x^{\mathbf{a}^{(l)}(j)}}{h_{l,z^{(l)}}(x)}dz_j^{(l)}\wedge\phi \tag{6}$$

[1] K. Nishitani, master thesis 2011 (in Japanese), Kobe University

The dual connection $\nabla^{\vee GM}:\mathcal{H}^{n\vee}_{dR}\to\Omega^1_Y(\mathcal{H}^{n\vee}_{dR})$ is defined by replacing γ_l by $-\gamma_l$ in (6).

The $\mathcal{D}_Y$-module structures of $\mathcal{H}^n_{dR}$ and $\mathcal{H}^{n\vee}_{dR}$ are compatible with the cohomology intersection form. Namely, for any sections $\phi\in\mathcal{H}^n_{dR}\restriction_U$ and $\psi\in\mathcal{H}^{n\vee}_{dR}\restriction_U$, we have

$$d_z\langle\phi,\psi\rangle_{ch}=\langle\nabla^{GM}\phi,\psi\rangle_{ch}+\langle\phi,\nabla^{\vee GM}\psi\rangle_{ch}. \tag{7}$$

We call (7) the secondary equation. Let us rewrite it in terms of local frames. Let $\{\phi_i\}_{i=1}^r$ (resp. $\{\psi_i\}_{i=1}^r$) be a free basis of $\phi\in\mathcal{H}^n_{dR}\restriction_U$ (resp. $\psi\in\mathcal{H}^{n\vee}_{dR}\restriction_U$). We set $I=I_{ch}=(\langle\phi_i,\psi_j\rangle)_{i,j}$ and call it the cohomology intersection matrix. On the other hand, there is a $r\times r$ matrix Ω (resp. $\Omega^\vee$) with values in 1-forms on U such that the connection ∇^{GM} (resp. $\nabla^{\vee GM}$) is trivialized as $d_z+\Omega\wedge$ (resp. $d_z+\Omega^\vee\wedge$). Then, the secondary equation is equivalent to the system

$$d_zI={}^t\Omega I+I\Omega^\vee. \tag{8}$$

We also call (8) the secondary equation. The theorem which our algorithm is based on is the following

Theorem 1. [16] *Under the regularization condition, any rational function solution I of the secondary equation (8) is, up to a scalar multiplication, equal to I_{ch}.*

2.3 GKZ system behind

In [16], it is discussed that Theorem 1 is true for more general direct image $\mathcal{D}$-module. However, by employing the combinatorial structure behind our integrable connection $\mathcal{H}^n_{dR}\restriction_U$, we can show that the cohomology intersection number in question has a rational expression with respect to z and δ.

Let us recall the definition of GKZ system ([8]). For a given $d\times N$ $(d<N)$ integer matrix $A=(\mathbf{a}(1)|\cdots|\mathbf{a}(N))$ and a parameter vector $\delta\in\mathbb{C}^d$, GKZ system $M_A(\delta)$ is defined as a system of partial differential equations on $\mathbb{C}^N$ given by

$$M_A(\delta):\begin{cases}E_i\cdot f(z)=0 & (i=1,\dots,d) & \text{(9a)}\\ \square_u\cdot f(z)=0 & \big(u\in\operatorname{Ker}(A\times:\mathbb{Z}^{N\times1}\to\mathbb{Z}^{d\times1})\big), & \text{(9b)}\end{cases}$$

where E_i and $\square_u$ for $u={}^t(u_1,\dots,u_N)$ are differential operators defined by

$$E_i=\sum_{j=1}^N a_{ij}z_j\frac{\partial}{\partial z_j}+\delta_i,\quad \square_u=\prod_{u_j>0}\left(\frac{\partial}{\partial z_j}\right)^{u_j}-\prod_{u_j<0}\left(\frac{\partial}{\partial z_j}\right)^{-u_j}. \tag{10}$$

For convenience, we assume an additional condition $\mathbb{Z}A\overset{def}{=}\mathbb{Z}\mathbf{a}(1)+\cdots+\mathbb{Z}\mathbf{a}(N)=\mathbb{Z}^d$. In our setting, we put $A_l=(\mathbf{a}^{(l)}(1)|\dots|\mathbf{a}^{(l)}(N_l))$, $d=n+k$, $N=N_1+\cdots+N_k$. We define an $(n+k)\times N$ matrix A by

$$A=\left(\begin{array}{c|c|c|c}1\cdots1 & 0\cdots0 & \cdots & 0\cdots0\\ \hline 0\cdots0 & 1\cdots1 & \cdots & 0\cdots0\\ \hline \vdots & \vdots & \ddots & \vdots\\ \hline 0\cdots0 & 0\cdots0 & \cdots & 1\cdots1\\ A_1 & A_2 & \cdots & A_k\end{array}\right). \tag{11}$$

We put $\delta=\begin{pmatrix}\gamma\\ c\end{pmatrix}$. It is known that GKZ system $M_A(\delta)$ is holonomic ([1]). Moreover, $\mathcal{H}^n_{dR}$ (resp. $\mathcal{H}^{n\vee}_{dR}$) is isomorphic to GKZ system $M_A(\delta)$ (resp. $M_A(-\delta)$) and the regularization condition is true when the parameter vector δ is non-resonant and $\gamma_l\notin\mathbb{Z}$ (see [9, 2.9] and [15, Theorem 2.12]). The isomorphism $M_A(\delta)\simeq\mathcal{H}^n_{dR}$ is given by the correspondence $[1]\mapsto[\frac{dx}{x}]$. Thus, any section $\phi\in\mathcal{H}^n_{dR}\restriction_U$ can be written as $\phi=P\cdot[\frac{dx}{x}]$ for some linear differential operator P.

Theorem 2. [16] *Suppose that A as in (11) admits a unimodular regular triangulation T and δ is non-resonant and $\gamma_l\notin\mathbb{Z}$. Then, for any $P_1,P_2\in\mathbb{Q}(\delta)\langle z,\partial_z\rangle$, the cohomology intersection number $\frac{\langle P_1\cdot\frac{dx}{x},P_2\cdot\frac{dx}{x}\rangle_{ch}}{(2\pi\sqrt{-1})^n}$ belongs to the field $\mathbb{Q}(\delta)(z)$.*

3 An algorithm of finding the Pfaffian system for a given basis

In this section, we set $\beta := -\delta$. With this notation, we put $H_A(\beta) := M_A(\delta)$. This is because we use some results from [12] and [23] where hypergeometric ideal is denoted by $H_A(\beta)$ while our main references [15], [16] denote it by $M_A(\delta)$.

Let ω_q be the differential form

$$\prod_{l=1}^{k} h_l^{-q'_l} x^{q''} \frac{dx}{x}, \quad q = (q', q'') \in \mathbb{Z}^k \times \mathbb{Z}^n \tag{12}$$

It is known that there exists a basis of the twisted cohomology group of which elements are of the form ω_q when δ is generic. Let $\{\omega_q \,|\, q \in Q\}$ be a basis of the twisted cohomology group. We will give an algorithm to find a Pfaffian system $\frac{\partial}{\partial z_i}\omega = P_i \omega$ with respect to this basis $\omega = (\omega_q \,|\, q \in Q)^T$. Note that algorithms to translate a given holonomic ideal to a Pfaffian system are well known (see, e.g., [12, Chap 6]). However, as long as we know, algorithms to find the Pfaffian system with respect to a given basis of twisted cohomology group do not appear in literature.

Put $\partial_i = \frac{\partial}{\partial z_i}$. In this subsection, we use $\bullet$ to denote the action to avoid a confusion with the multiplication. The function $\langle \omega_q \rangle$ is a solution of the hypergeometric system $H_A(\beta - q)$. The main point of our method is of use of the following contiguity relation

$$\frac{1}{\mathbf{a}'_i \cdot (\beta - q)} \partial_i \bullet \langle \omega_q \rangle = \langle \omega_{q'} \rangle, \quad q' = q + \mathbf{a}_i \tag{13}$$

where $\mathbf{a}_i$ is the i-th column vector of A and $\mathbf{a}'_i$ is the column vector that the first k elements are equal to those of $\mathbf{a}_i$ and the last n elements are 0. For example, $\mathbf{a}'_1 = (1, 0, \ldots, 0)$, $\mathbf{a}'_2 = (1, 0, \ldots, 0)$, …, $\mathbf{a}'_{N_1+1} = (0, 1, 0, \ldots, 0)^T$, …. The relation (13) can be proved by differentiating $\langle \omega_q \rangle = \int_\Gamma h_1^{-\gamma_1 - q'_1} \cdots h_k^{-\gamma_k - q'_k} x^{c+q''} \frac{dx}{x}$, with respect to z_i where we have $\beta - q = (-\gamma_1 - q'_1, \ldots, -\gamma_k - q'_k, -c_1 - q''_1, \ldots, -c_n - q''_n)^T$.

In [23, Algorithm 3.2], an algorithm to obtain the operator C_i satisfying

$$C_i \partial_i - b_i(\beta) = 0 \mod H_A(\beta) \tag{14}$$

is given. The polynomial b_i is a b-function in the direction i [23, Th 3.2]. Note that the algorithm outputs the operator C_i in $\mathbb{C}\langle z_1, \ldots, z_N, \partial_1, \ldots, \partial_N \rangle$, which does not depend on the parameter β. Since $\langle \omega_q \rangle$ is a solution of $H_A(\beta - q)$, we have the following inverse contiguity relation

$$\frac{\mathbf{a}'_i \cdot (\beta - q'')}{b_i(\beta - q'')} C_i \bullet \langle \omega_q \rangle = \langle \omega_{q''} \rangle, \quad q'' = q - \mathbf{a}_i. \tag{15}$$

Example 1. (Gauss hypergeometric function ${}_2F_1$.) Put

$$A = \begin{pmatrix} 1\ 1 & 0\ 0 \\ \hline 0\ 0 & 1\ 1 \\ \hline 0\ 1 & 0\ 1 \end{pmatrix} \tag{16}$$

Then, $h_1 = z_1 + z_2 x$, $h_2 = z_3 + z_4 x$. We have

$$\langle \omega_{(1,0,0)} \rangle = \int_\Gamma h_1^{-\gamma_1} h_2^{-\gamma_2} x^c \frac{1}{h_1} \frac{dx}{x}. \tag{17}$$

We can show that $\{\omega_{(1,0,0)}, \omega_{(1,0,0)} - \omega_{(0,1,0)}\}$ is a basis of the twisted cohomolgy group. This A is normal and the b-function $b_4(s) \in \mathbb{Q}[s_1, s_2, s_3]$ for the direction z_4 is $b_4(s) = s_2 s_3$. Then, $C_4 = z_2 z_3 \partial_1 + (\theta_2 + \theta_3 + \theta_4) z_4$ where $\theta_i = z_i \partial_i$ by reducing $(\theta_3 + \theta_4)(\theta_2 + \theta_4)$ by the toric ideal $I_A = \langle \underline{\partial_2 \partial_3} - \partial_1 \partial_4 \rangle$ (see Algorithm 3.2 of [23]).

Our algorithm to find a Pfaffian system with respect to a given basis of the twisted cohomology group is as follows.

Algorithm 1.
Input: $\{\omega_q \,|\, q \in Q\}$, a basis of the twisted cohomology group. A direction (index) i.
Output: P_i, the coefficient matrix of the Pfaffian system $\partial_i - P_i$.

1. Compute a Gröbner basis G of $H_A(\beta)$ in the ring of differential operators with rational function coefficients. Let S be a column vector of standard monomials with respect to G.
2. Put
$$F(Q) = (F(q) \mid q \in Q)^T, \quad F(q) = \prod_{r_i<0} C_i^{-r_i} \prod_{r_i>0} \partial_i^{r_i} \frac{1}{BB'}, \quad q = \sum_{i=1}^{N} r_i \mathbf{a}_i \tag{18}$$
It is a vector of which elements are in the ring of differential operators and the order of product is $i = N, N-1, \ldots, 3, 2, 1$. In other words, we apply operators from ∂_1. The polynomial B comes from the coefficient of the contiguity relation (15) and is equal to
$$B = \prod_{j=1, r_j<0}^{N} \frac{b_j(\beta'_j + \mathbf{a}_j)}{\mathbf{a}'_j \cdot (\beta'_j + \mathbf{a}_j)} \frac{b_j(\beta'_j + 2\mathbf{a}_j)}{\mathbf{a}'_j \cdot (\beta'_j + 2\mathbf{a}_j)} \cdots \frac{b_j(\beta'_j + (-r_j)\mathbf{a}_j)}{\mathbf{a}'_j \cdot (\beta'_j + (-r_j)\mathbf{a}_j)}, \tag{19}$$
$$\beta'_j = \beta - \sum_{r_l>0} r_l \mathbf{a}_l + \sum_{l=1, r_l<0}^{j-1} (-r_l)\mathbf{a}_l \tag{20}$$
The polynomial B' comes from the denominator of the contiguity relation (13) and is equal to
$$B' = \prod_{j=1, r_j>0}^{N} \left(\mathbf{a}'_j \cdot (\beta'_j)\right) \left(\mathbf{a}'_j \cdot (\beta'_j - \mathbf{a}_j)\right) \cdots \left(\mathbf{a}'_j \cdot (\beta'_j - (r_j - 1)\mathbf{a}_j)\right), \tag{21}$$
$$\beta'_j = \beta - \sum_{r_l>0, l<j} r_l \mathbf{a}_l \tag{22}$$
3. Compute the normal form of the vector $\partial_i F(Q)$ and $F(Q)$. Write the normal forms of them as $P'S$ and $P''S$ respectively where P' and P'' are matrices with rational function entries.
4. Output $P_i = P'(P'')^{-1}$.

The matrix P'' is invertible if and only if the given set of differential forms $\{\omega_q\}$ is a basis of the twisted cohomology group.

We show the correctness of the algorithm. Take an element $q \in Q$. We express $\langle \omega_q \rangle$ in terms of $\langle \omega_0 \rangle$, which is a solution of $H_A(\beta)$, by the contiguity relations (13) and (15). Note that the contiguity relations for functions $\langle \omega_q \rangle$ give the contiguity relations for cohomology classes $[\omega_q]$ by virtue of the perfectness of the pairing between the twisted homology and the twisted cohomology groups. The point of the correctness is the following identity
$$F(q) \bullet \omega_0 = \omega_q. \tag{23}$$
Let us illustrate how to prove (23) by examples. We assume that $q = 2\mathbf{a}_1 + \mathbf{a}_2$ and $N_1 \geq 2$. Then ω_q can be obtained by applying (13) with $i = 1$ for two times and that with $i = 2$. We have
$$\omega_{\mathbf{a}_1} = \frac{1}{\mathbf{a}'_1 \cdot \beta} \partial_1 \bullet \omega_0 \tag{24}$$
$$\omega_{2\mathbf{a}_1} = \frac{1}{\mathbf{a}'_1 \cdot (\beta - \mathbf{a}_1)} \partial_1 \bullet \omega_{\mathbf{a}_1} \tag{25}$$
$$\omega_{2\mathbf{a}_1 + \mathbf{a}_2} = \frac{1}{\mathbf{a}'_2 \cdot (\beta - 2\mathbf{a}_1)} \partial_2 \bullet \omega_{2\mathbf{a}_1} \tag{26}$$
Thus, we obtain the numbers (21) and then (23). Let us consider the case that $q = -2\mathbf{a}_1 - \mathbf{a}_2$ and $N_1 \geq 2$. Then ω_q can be obtained by applying (15) with $i = 1$ for two times and that with $i = 2$. Since $\langle \omega_{-\mathbf{a}_1} \rangle$ is a solution of $H_A(\beta + \mathbf{a}_1)$, we have
$$[c_1 \partial_1 - b_1(\beta + \mathbf{a}_1)] \bullet \omega_{-\mathbf{a}_1} = 0 \tag{27}$$
from [23]. Then, we have
$$\omega_{-\mathbf{a}_1} = \frac{\mathbf{a}'_1 \cdot (\beta + \mathbf{a}_1)}{b_1(\beta + \mathbf{a}_1)} c_1 \bullet \omega_0 \tag{28}$$
$$\omega_{-2\mathbf{a}_1} = \frac{\mathbf{a}'_1 \cdot (\beta + 2\mathbf{a}_1)}{b_1(\beta + 2\mathbf{a}_1)} c_1 \bullet \omega_{-\mathbf{a}_1} \tag{29}$$

$$\omega_{-2\mathbf{a}_1-\mathbf{a}_2} = \frac{\mathbf{a}_2' \cdot (\beta + 2\mathbf{a}_1 + \mathbf{a}_2)}{b_2(\beta + 2\mathbf{a}_1 + \mathbf{a}_2)} c_2 \bullet \omega_{-2\mathbf{a}_1} \tag{30}$$

Thus, we obtain the numbers (19) and then (23). The general case can be shown by repeating these procedures and we can show that $F(q) \bullet \omega_0 = \omega_q$. When the normal form $F(q)$ with respect to the Gröbner basis G is $\sum p_i'' s_i$ where $S = (s_i)$ and p_i'' is a rational function in z and β, we have $\omega_q = F(q) \bullet \omega_0 = \sum p_i'' s_i \bullet \omega_0$. The correctness of the last two steps follows from this fact.

Example 2. This is a continuation of Example 1. We have $(1,0,0)^T = \mathbf{a_1}$ and $(0,1,0)^T = \mathbf{a_3}$. Then, the basis of the twisted cohomology group $F(Q)$ is expressed as $F(Q) = (\partial_1/\beta_1, \partial_1/\beta_1 - \partial_3/\beta_2)^T$ and $\partial_4 F(Q) = (\partial_4\partial_1/\beta_1, \partial_4\partial_1/\beta_1 - \partial_4\partial_3/\beta_2)^T$. We can obtain a Gröbner basis with the set of the standard monomials is $\{\partial_4, 1\}$. We multiply $\beta_1\beta_2$ to $F(Q)$ and $\partial_4 F(Q)$ in order to avoid rational polynomial arithmetics. Then, the normal form, for example, of $\beta_2\partial_1$ is $\frac{1}{z_1z_4 - z_2z_3}\left((\beta_1(\beta_1+\beta_2)z_4)\partial_4 - \beta_2^2\beta_3\right)$. By computing the other normal forms, we obtain the matrix

$$P_4 = \begin{pmatrix} \frac{-\beta_2(z_3-z_1)}{z_1z_4-z_2z_3} & \frac{\beta_2 z_3}{z_1z_4-z_2z_3} \\ \frac{-((\beta_2z_3+(-\beta_2+\beta_3)z_1)z_4+(\beta_1-\beta_3)z_2z_3-\beta_1z_1z_2)}{z_4(z_1z_4-z_2z_3)} & \frac{(\beta_2z_3+\beta_3z_1)z_4+(\beta_1-\beta_3)z_2z_3}{z_4(z_1z_4-z_2z_3)} \end{pmatrix}. \tag{31}$$

4 Implementation and examples

We implemented our algorithms on the computer algebra system Risa/Asir [21] with a Polymake interface. Polymake (see, e.g., [20], [22]) is a system for polyhedral geometry and it is used for efficient computation of contiguity relations ([23, Algorithm 3.2]). Here is an input[2] to find the coefficient matrix P_4 for Example 1 with respect to the variable z_4 and $z_1 = z_2 = z_3 = 1$ (note that in our implementation x is used instead of z).

```
P4=pfaff_eq(A=[[1,1,0,0],[0,0,1,1],[0,1,0,1]],
            Beta=[-gamma1,-gamma2,-c],
            Ap = [[1,1,0,0],[0,0,1,1],[0,0,0,0]],
            Rvec = [[1,0,0,0],[0,0,1,0]],DirX=[dx4] //Rvec is the set of r's in Algorithm 1.
     | xrule=[[x1,1],[x2,1],[x3,1]],
       cg=matrix_list_to_matrix([[1,0],[1,-1]]));//get Pfaffian sys for cg*(the basis omega_q)
```

It outputs the following coefficient matrix

$$P_4 = \begin{pmatrix} 0 & \frac{-\gamma_2}{x_4-1} \\ \frac{c}{x_4} & \frac{(-c-\gamma_2)x_4+c-\gamma_1}{(x_4-1)x_4} \end{pmatrix} \tag{32}$$

Example 3. (${}_3F_2$, see, e.g., [24, p.224], [19].) Let $A = \left(\begin{array}{cc|cc|cc} 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$. The integrals are

$$\int_\Gamma (z_1x_1 + z_2)^{-\gamma_1}(z_3x_2 + z_4x_1)^{-\gamma_2}(z_5 + z_6x_2)^{-\gamma_3} x_1^{c_1} x_2^{c_2} \omega_i \tag{33}$$

where

$$\omega_1 = \frac{dx_1dx_2}{(z_1x_1 + z_2)x_1x_2}, \omega_2 = \frac{dx_1dx_2}{(z_5 + z_6x_2)x_1x_2}, \omega_3 = \frac{dx_1dx_2}{(z_3x_2 + z_4x_1)x_1x_2} \tag{34}$$

When $z_2 = -1, z_3 = z_4 = z_5 = z_6 = 1$, the coefficient matrix for z_1 for the basis $(\langle\omega_1\rangle, \langle\omega_2\rangle, \langle\omega_3\rangle)^T$ is

$$P_1 = \begin{pmatrix} \frac{\beta_4z_1+\beta_2+\beta_3+\beta_4+\beta_5}{z_1(z_1-1)} & \frac{\beta_3(\beta_4-\beta_1-\beta_2)}{\beta_1z_1(z_1-1)} & \frac{(-\beta_4+1)\beta_2(-\beta_2+\beta_4+\beta_5+1)}{\beta_4\beta_1z_1(z_1-1)} \\ \frac{(\beta_2+\beta_3-\beta_5)\beta_1}{\beta_3(z_1-1)} & \frac{\beta_1z_1+\beta_2-\beta_4}{z_1(z_1-1)} & \frac{(-\beta_4+1)\beta_2(-\beta_2+\beta_4+\beta_5+1)}{\beta_4\beta_3z_1(z_1-1)} \\ \frac{\beta_4(\beta_2+\beta_3-\beta_5)\beta_1}{(-\beta_4+1)\beta_2(z_1-1)} & \frac{\beta_4\beta_3(\beta_1+\beta_2-\beta_4)}{(-\beta_4+1)\beta_2(z_1-1)} & \frac{(-\beta_2+\beta_4+\beta_5+1)}{z_1-1} \end{pmatrix} \tag{35}$$

The result can be obtained in a few seconds.

[2] The algorithm 1 is implemented in `saito-b.rr` distributed at [25]

5 An algorithm of finding the cohomology intersection matrix

Theorem 3. [16] *Given a matrix $A = (a_{ij})$ as in (11) admitting a unimodular regular triangulation T. When parameters are non-resonant, $\gamma_l \notin \mathbb{Z}$ and moreover the set of series solutions by T is linearly independent, the intersection matrix of the twisted cohomology group of the GKZ system associated to the matrix A can be algorithmically determined.*

We denote by Ω_i the coefficient matrix of Ω with respect to the 1-form dz_i. The algorithm we propose is summarized as follows.

Algorithm 2. (A modified version of the algorithm in [16].)
Input: Free bases $\{\phi_j\}_j \subset \mathcal{H}^n_{dR} \restriction_U$, $\{\psi_j\}_j \subset \mathcal{H}^{n\vee}_{dR} \restriction_U$ which are expressed as (12).
Output:The secondary equation (8) and the cohomology intersection matrix $I_{ch} = (\langle \phi_i, \psi_j \rangle_{ch})_{i,j}$.

1. *Obtain a Pfaffian system with respect to the given bases $\{\phi_j\}_j$ and $\{\psi_j\}_j$, i.e., obtain matrices $\Omega_i = (\omega_{ijk})$ and $\Omega_i^\vee = (\omega^\vee_{ijk})$ so that the equalities*
$$\partial_i \phi_j = \sum_k \omega_{ikj} \phi_k, \quad \partial_i \psi_j = \sum_k \omega^\vee_{ikj} \psi_k \tag{36}$$
hold by Algorithm 1.
2. *Find a non-zero rational function solution I of the secondary equation*
$$\partial_i I - {}^t\Omega_i I - I \Omega_i^\vee = 0, \quad i = 1, \ldots, N. \tag{37}$$
To be more precise, see, e.g., [5], [4],[18] and their references.
3. *Determine the scalar multiple of I by [15, Theorem 8.1].*

Example 4. This is a continuation of Example 3. We want to evaluate the cohomology intersection matrix $I_{ch} = (\langle \omega_i, \omega_j \rangle_{ch})^3_{i,j=1}$. By solving the secondary equation (for example, using [5]), we can verify that $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$ entries of I_{ch} are all independent of z_1. Therefore, we can obtain the exact values of these entries by taking a unimodular regular triangulation $T = \{23456, 12456, 12346\}$ and substituting $z_1 = 0$ in [15, Theorem 8.1]. Thus, we get a correct normalization of I_{ch} and the matrix $\frac{I_{ch}}{(2\pi\sqrt{-1})^2}$ is given by

$$\begin{bmatrix} r_{11} & \frac{\beta_4+\beta_5}{\beta_5\beta_4(\beta_2-\beta_4-\beta_5)} & \frac{\beta_4(\beta_1+\beta_2-\beta_4-\beta_4)z_1-\beta_5\beta_3}{\beta_5(\beta_4+1)(\beta_2-\beta_4-\beta_5)(\beta_2-\beta_4-\beta_5+1)} \\ \frac{\beta_4+\beta_5}{\beta_5\beta_4(\beta_2-\beta_4-\beta_5)} & r_{22} & \frac{-(\beta_4\beta_1 z_1-\beta_5\beta_2-\beta_5\beta_3+\beta_5\beta_4+\beta_5^2)}{\beta_5(\beta_4+1)(\beta_2-\beta_4-\beta_5)(\beta_2-\beta_4-\beta_5+1)} \\ \frac{\beta_4(\beta_1+\beta_2-\beta_4-\beta_5)z_1-\beta_5\beta_3}{\beta_5(\beta_4-1)(\beta_2-\beta_4-\beta_5)(\beta_2-\beta_4-\beta_5-1)} & \frac{-\beta_4\beta_1 z_1+\beta_5(\beta_2+\beta_3-\beta_4-\beta_5)}{\beta_5(\beta_4-1)(\beta_2-\beta_4-\beta_5)(\beta_2-\beta_4-\beta_5-1)} & r_{33} \end{bmatrix} \tag{38}$$

where
$$r_{11} = -\frac{(\beta_4\beta_2 + (\beta_4+\beta_5)\beta_3)\beta_1 + \beta_4\beta_2^2 + (\beta_4\beta_3 - \beta_4^2 - \beta_5\beta_4)\beta_2 + (-\beta_4^2 - \beta_5\beta_4)\beta_3}{\beta_5\beta_4\beta_1(\beta_2-\beta_4-\beta_5)(\beta_2+\beta_3-\beta_5)} \tag{39}$$
$$r_{22} = -\frac{(\beta_5\beta_2 + (\beta_4+\beta_5)\beta_3 - \beta_5\beta_4 - \beta_5^2)\beta_1 + \beta_5\beta_2^2 + (\beta_5\beta_3 - \beta_5\beta_4 - \beta_5^2)\beta_2}{\beta_5\beta_4\beta_3(\beta_2-\beta_4-\beta_5)(\beta_1+\beta_2-\beta_4)} \tag{40}$$
$$r_{33} = -\frac{\beta_4\left\{(\beta_1\beta_2-\beta_1\beta_5+{\beta_2}^2-\beta_2\beta_4-2\beta_2\beta_5+\beta_4\beta_5+{\beta_5}^2)\beta_1\beta_4{z_1}^2-2\beta_1\beta_3\beta_4\beta_5 z_1+({\beta_2}^2+\beta_2\beta_3-2\beta_2\beta_4-\beta_2\beta_5-\beta_3\beta_4+{\beta_4}^2+\beta_4\beta_5)\beta_3\beta_5\right\}}{\beta_5\beta_2(\beta_4-1)(\beta_4+1)(\beta_2-\beta_4-\beta_5)(\beta_2-\beta_4-\beta_5-1)(\beta_2-\beta_4-\beta_5+1)} \tag{41}$$

Example 5. Let $A = \begin{pmatrix} 1\,1\,1\,1\,1 \\ \hline 0\,1\,0\,2\,0 \\ 0\,0\,1\,0\,2 \end{pmatrix}$. The integrals are
$$\int_\Gamma h_1^{-\gamma_1} x_1^{c_1} x_2^{c_2} \omega_i, \quad h_1 = z_1 + z_2 x_1 + z_3 x_2 + x_4 x_1^2 + z_5 x_2^2 \tag{42}$$
where
$$\omega_1 = \frac{dx_1 dx_2}{x_1 x_2}, \omega_2 = x_1\omega_1 = \frac{dx_1 dx_2}{x_2}, \omega_3 = x_2^2\omega_1 = \frac{x_2 dx_1 dx_2}{x_1}, \omega_4 = x_1 x_2 \omega_1 = dx_1 dx_2. \tag{43}$$

Note that this A is not normal. When $z_1 = z_4 = z_5 = 1$, we have obtained the coefficient matrices P_2 and P_3 in about 9 hours 45 minutes on a machine with Intel(R) Xeon(R) CPU E5-4650 2.70GHz and 256GB memory. The $(1,1)$ element of P_2 is

$$\frac{((b_2 z_2^2 + b_{123}) z_3^2 + b_2 z_2^4 + b_{132} z_2^2 - 32b_1 + 16b_2 + 16b_3 - 16)}{z_2(z_2-2)(z_2+2)(z_3^2+z_2^2-4)} \tag{44}$$

where $b_1 = -\gamma_1$, $b_2 = -c_1$, $b_3 = -c_2$ and $b_{ijk} = 8b_i - 4b_j - 8b_k + 4$. A complete data of P_2 and P_3 is at [25]. The intersection matrix can be obtained by [5] in a few second when we specialize b_i's to rational numbers. See [25] as to Maple inputs for it.

References

1. A.Adolphson, Hypergeometric functions and rings generated by monomials, Duke Math. J., 73 (1994), 269–290.
2. K.Aomoto and M.Kita, Theory of hypergeometric functions, (in Japanese) Springer-Tokyo, 1994.
3. K.Aomoto, M.Kita, Theory of hypergeometric functions, Springer-Verlag, Tokyo, 2011. (English translation of [2])
4. M. Barkatou. On rational solutions of systems of linear differential equations, Journal of Symbolic Computation, 28 (1999), 547–567.
5. M. Barkatou, T. Cluzeau, C. El Bacha, J.-A. Weil, IntegrableConnections — A Maple package for computing closed form solutions of integrable connections, `https://www.unilim.fr/pages_perso/thomas.cluzeau/Packages/IntegrableConnections/PDS.html`
6. K.Cho and K.Matsumoto, Intersection theory for twisted cohomologies and twisted Riemann's period relations I, Nagoya Mathematical Journal, 139 (1995), 67–86.
7. P.Deligne, Équations différentielles à points singuliers réguliers, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin-New York, 1970.
8. I.M.Gel'fand, M.M.Kapranov, and A.V.Zelevinsky, Hypergeometric functions and toric varieties. (Russian) Funktsional. Anal. i Prilozhen. 23 (1989), no. 2, 1226; translation in Functional Analysis and Applications 23 (1989), no. 2, 94106
9. I.M.Gel'fand, M.M.Kapranov, and A.V.Zelevinsky, Generalized Euler integrals and A-hypergeometric functions, Advances in Mathematics, 84 (1990), 255–271.
10. Y.Goto, J.Kaneko, K.Matsumoto, Pfaffian of Appell's hypergeometric system F_4 in terms of the intersection form of twisted cohomology groups, Publication of Research Institute of Mathematical Sciences, 52 (2016), 223–247.
11. Y.Goto, , K.Matsumoto, Pfaffian equations and contiguity relations of the hypergeometric function of type $(k+1, k+n+2)$ and Their Applications, Funkcialaj Ekvacioj, 61 (2018), 315–347.
12. Edited by T.Hibi, Gröbner bases: statistics and software systems, 2013, Springer.
13. T.Hibi, K.Nishiyama, N.Takayama, Pfaffian systems of A-hypergeometric equations I: Bases of twisted cohomology groups, Advances in Mathematics, 306 (2017), 303–327.
14. M.Kita, M.Yoshida, Intersection theory for twisted cycles, Mathematische Nachrichten, 166 (1994), 287-304.
15. S.-J.Matsubara-Heo, Euler and Laplace integral representations of GKZ hypergeometric functions, arXiv:1904.00565
16. S.-J.Matsubara-Heo, N.Takayama, An algorithm of computing cohomology intersection number of hypergeometric integrals, arXiv:1904.01253
17. K.Matsumoto, Intersection numbers for logarithmic k-forms, Osaka J. Math., 35 (1998), 873–893.
18. T.Oaku, N.Takayama, H.Tsai, Polynomial and rational solutions of holonomic systems. J. Pure and Applied Algebra, 164 (2001), 199–220.
19. K.Ohara, Y.Sugiki, N.Takayama, Quadratic relations for generalized hypergeometric functions ${}_pF_{p-1}$, Funkcialaj Ekvacioj, 46 (2003), 213–251.
20. Polymake, `https://polymake.org`
21. Risa/Asir, a computer algebra system. `http://www.math.kobe-u.ac.jp/Asir`, `http://www.openxm.org`
22. B. Assarf, E. Gawrilow, K. Herr, M. Joswig, B. Lorenz, A. Paffenholz, T. Rehn. Computing convex hulls and counting integer points with polymake. Mathematical Programming Computation, 9 (2017), 1-38.
23. M.Saito, B.Sturmfels, N.Takayama, Hypergeometric polynomials and integer programming, Compositio Mathematica, 115 (1999), 185–204.
24. M.Saito, B.Sturmfels, N.Takayama, Gröbner deformations of hypergeometric differential equations, Springer, 2000.
25. Rational function solutions and intersection numbers, `http://www.math.kobe-u.ac.jp/OpenXM/Math/intersection2`