**Paper**

*Shift relations and reducibility of some Fuchsian differential equations of order* $2,\dots,6$ *with three singular points* by Akihito Ebisu, Yoshishige Haraoka, Hiroyuki Ochiai, Takeshi Sasaki and Masaaki Yoshida, submitted

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**Explanation of data**

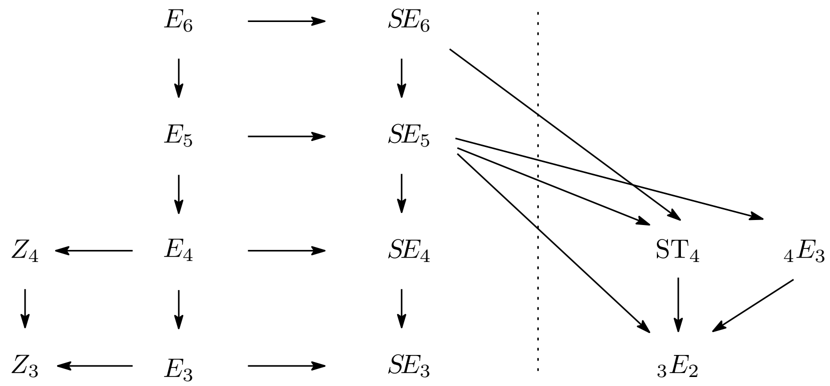
For the Fuchsian differential equations treated in the paper we list the shift operators in the form that the data is readable by the help of the mathematical software Maple.

**Definition of the differential equations**

The equations we treated in the paper are

$$E_6,\ SE_6;\quad E_5,\ SE_5;\quad E_4,\ SE_4,\ Z_4,\ \mathrm{ST}_4,\ {}_4E_3;\quad E_3,\ SE_3,\ Z_3,\ E_{3a},\dots,E_{3d},\ {}_3E_2;\quad E_2.$$

The equation $E_2$ is the Gauss hypergeometric equation, which is related to all others; They are mutually related as in the figure




Horizontal arrows stand for specializations keeping the spectral type, and other lines for factorizations. The three equations on the right side of the dotted line are rigid, while each of the remaining ones has one accessory parameter. The four equations $E_{3a},\dots,e_{3d}$ and some arrows are not shown to make the figure less complex.

In the following, we use a maple notation $dx$ used in *DEtools* of Maple, which means the differential operator $d/dx$ relative to the variable $x$ and the notation $z$ denotes the operator $x*dx$. The mark $*$ means the multiplication as well as composition of differential operators.

We summarize the defintion of the differential equations. The text-file **equations.txt** lists the maple forms of the differential equations.

- $E_6 = E_6(e1,\dots,e9) = T0 + T1*dx + T2*dx^2 + T3*dx^3$

  ```
  T0:=(z+s+2)*(z+s+1)*(z+s)*(z+e7)*(z+e8)*(z+e9):
  T1:=(z+s+2)*(z+s+1)*B1:
  T2:=(z+s+2)*B2:
  T3:=-(z+3-e1)*(z+3-e2)*(z+3-e3):
  B1:=T13*z^3+T12*z^2+T11*z+T10:
  B2:=T23*z^3+T22*z^2+T21*z+T20:
  ```

  where $s$ is determined by the Fuchs relation $e1+\cdots+e9+3s+9=15$.

  The operators $B1$, $B2$ and the polynomials $Tij$ are given in the file **equations.txt**.

- $SE_6$ is a specialization of $E_6$ with the condition

  $$e1-2e2+e3 = e4-2e5+e6 = e7-2e8+e9.$$

  It is parameterized by $(a,b,c,g,p,q,r)$ by the relations

```
e1:=p+r+1:
e2:=a+c+p+r+2:
e3:=2a+2c+g+p+r+3:
e4:=q+r+1:
e5:=b+c+q+r+2:
e6:=2b+2c+g+q+r+3:
e7:=-2c-p-q-r-1:
e8:=-a-b-2c-p-q-r-g-2:
e9:=-2a-2b-2c-p-q-r-g-3:
s:=-r:
```

- The equation $E_5 = E_5(e1, \ldots, e8) = x * Pn + P0 + P1 * dx + P2 * dx^2$ is the quotient of $E_6$ with the condition $e9 = 0$ divied by $dx$ on the right.

```
Pn:=(z-r+1)*(z-r+2)*(z-r+3)*(z+e7+1)*(z+e8+1):
P0:=(z-r+1)*(z-r+2)*B1(e9=0):
P1:=(z-r+2)*B2(e9=0):
P2:=-(z+3-e1)*(z+3-e2)*(z+3-e3):
```

where $r = (e1 + \cdots + e8 - 6)/3$.

- $SE_5$ is a specialization of $E_5$ with the condition

$$e1 - 2e2 + e3 = e4 - 2e5 + e6 = e7 - 2e8.$$

It is parameterized by $(a, b, c, g, p, q)$ as

```
e1:= -2*a - 2*b - 2*c - g - q - 2:
e2:= -a - 2*b - c - g - q - 1:
e3:= -2*b - q:
e4:= -2*a - 2*b - 2*c - g - p - 2:
e5:= -b  - 2*a- c - g - p - 1:
e6:= -2*a - p:
e7:= 2*a + 2*b + g + 2:
e8:= a + b + 1:
```

- $E_4 = E_4(e1, \ldots, e7) = Q_0 + Q_1 * dx + Q_2 * dx^2$ is defined as

```
Q0:=(z+e5)*(z+e6)*(z+e7)*(z+e8):
Q1:=-2*z^3+Q12*z^2+Q11*z+Q10:
Q12:=e1+e2-e5-e6-e7-e8-5:
Q11:=3*(e1+e2)-e1*e2+e3*e4-e5*e6-e5*e7-e5*e8-e6*e7-e6*e8-e7*e8-8:
Q2:=(z-e1+2)*(z-e2+2):
```

where $e8$ is determined by the Fuchs raltion $e1 + e2 + \cdots + e7 + e8 = 4$ and $Q10$ is given in **equations.txt**. The equation is written also as

$$E_4 := x^2 * (x-1)^2 dx^4 + p3 * dx^3 + p2 * dx^2 + p1 * dx + p0 :$$

where

```
p3:= x*(x-1)*((-t11-t12+10)*x+t11-5):
p2:= (-3*t11-3*t12+t23+19)*x^2 +(5*t11+t12-t21+t22-t23-19)*x+4-2*t11+t21:
p1:= (t3-t11-t12+t23+5)*x+Q10:
p0:= e5*e6*e7*e8:
```

Refer to **equations.txt** for $t11$, $t12$, ....

- $SE_4$ is a specialization of $E_4$ with the condition

$$e1-2e2+1=e3-2e4+1=e5-2e6+e7+e8,\quad e1+\cdots+e7+e8=4.$$

  It is parameterized by $(a,b,c,g,q)$ as

```
e1:=-2b-q-1:
e2:=-a-2b-c-g-q-2:
e3:=-2c-q-1:
e4:=-a-2c-b-g-q-2:
e5:=q+1:
e6:=b+c+q+2:
e7:=2b+2c+g+q+3:
e8:=2a+2b+2c+g+q+4:
```

- $Z_4$ is a specialization of $E_4$ with the condition

$$e1+e2-1=e3+e4-1=-(e5+e6-1).$$

  It is parameterized by $(A0,A1,A2,A3,k)$ as

```
e1=1/2-A0-k:
e2=1/2+A0-k:
e3=1/2-A1-k:
e4=1/2+A1-k:
e5=1/2-A2+k:
e6=1/2+A2+k:
e7=1/2-A3+k:
e8=1/2+A3+k:
```

- $\mathrm{ST}_4=\mathrm{ST}_4(e1,\ldots,e6)=V0+V1*dx+V2*dx^2$ is given as

```
V0 := (z+s+1)*(z+s)*(z+e5)*(z+e6):
V1 := (z+s+1)*(-2*z^2+(e1+e2-e5-e6-4)*z
        +1/4*((e6-e5)^2-(e3-e4)^2+(e1-e2)^2+2*(e1+e2-3)*(e5+e6+1)-1)):
V2 := (z+2-e1)*(z+2-e2):
```

  where $s$ is determined by the Fuchs relation $e1+e2+e3+e4+e5+e6+2*s+3=6$.

- ${}_4E_3(a0,a1,a2,a3;b1,b2,b3):=(z+a0)*(z+a1)*(z+a2)*(z+a3)-(z+b1)*(z+b2)*(z+b3)*dx$ is the generalized hypergeometric equation of rank 4.

- $E_3=E_3(e1,\ldots,e6)=x*Sn+S0+S1*dx$ is defined as

```
Sn:=(z+e5)*(z+e6)*(z+e7):
S0:= -2*z^3+(2*e1+2*e2+e3+e4-3)*z^2
   +(-e1*e2+(e3-1)*(e4-1)-e5*e6-(e5+e6)*e7)*z+a00,\\
S1:=(z-e1+1)*(z-e2+1):
```

  where $e7$ is determined by the relation $e1+e2+e3+e4+e5+e6+e7=3$ and $a00$ is the accessory parameter defined as

```
54*a00:= -4*(e1+e2-e3-e4)^3-27*e5*e6*e7+9*(e1+e2-e3-e4)*(e5*e6+e5*e7+e6*e7-2)
    +9*e1*e2*(e1+e2-1)+18*(e1+e2-1)*(e3^2+e3*e4+e4^2)
    -9*e3*e4*(e3+e4-1)-18*(e3+e4-1)*(e1^2+e1*e2+e2^2):
```

- $SE_3$ is a specialization of $E_3$ with the condition

$$2e1-e2=2e3-e4=-e5+2e6-e7.$$

  It is parameterized by $(a,b,c,g)$ as:

```
e1:=a+c+1:
e2:=2e1+g:
e3:=b+c+1:
e4:=2e3+g:
e5:=-2c:
e6:=-(a+b+2c+g+1):
e7:=2e6+g-e5:
```

The accessory parameter turns out to be

$$a00 = c * (2 * a + 2 * c + 1 + g) * (2 * a + 2 * b + 2 * c + 2 + g).$$

- $Z_3$ is a specialization of $E_3$ with the condition

  $$e1 + e2 + e5 = e3 + e4 + e5 = 1.$$

  It is parameterized by $(A0, A1, A2, A3)$ as

```
e1:=-A0-A3:
e2:=A0-A3:
e3:=-A1-A3:
e4:=A1-A3:
e5:=2*A3+1:
e6:=A2+A3+1:
```

  The accessory parameter is given by

  $$a00 := (2 * A3 + 1) * (A0^2 - A1^2 - A3^2 + A2^2 - 2 * A3 - 1)/2.$$

- $E_{3a}$, $E_{3b}$, $E_{3c}$, $E_{3d}$ are specializations of $E_3$ with the conditions

  $$e3 = e1, e4 = e2;$$
  $$e2 = -e3 - e5 - 2 * e6 + 3 - e1, e4 = e3 - e5 + e6;$$
  $$e2 = 2 * e1 + e3 + e4, e5 = 1 - e1 - e3 - e4;$$
  $$e2 = 3/2 - e1 - 1/2 * e3 - 1/2 * e4 - 3/2 * e6, e5 = e1 + e6;$$

  respectively.

- ${}_3E_2(a0, a1, a2; b1, b2) := (z + a0) * (z + a1) * (z + a2) - (z + b1) * (z + b2) * dx$ is the generalized hypergeometric equation of rank 3.

- $E_2 := E_2(e1, e2, e3, e4) = E(a, b, c) = (z + a) * (z + b) - (z + c) * dx$ is the Gauss hypergeometric equation: We used the parameters $(e1, e2, e3, e4)$ defined as $e1 = 1 - c$, $e2 = c - a - b$, $e3 = a$, and $e4 = b$, where $e1 + e2 + e3 + e4 = 1$.

**Shift operators**

We review definitions of shift operators and explain the text-files for such operators.

Given a differential operator $E(a)$ with parameter $a$ of order $n$, suppose a shift operator $P_{a+}$ (*resp.* $P_{a-}$) exists, which is an operator of order lower than $n$ sending $\mathrm{Sol}(E(a))$ to $\mathrm{Sol}(E(a+1))$ (*resp.* $\mathrm{Sol}(E(a-1))$), we have a shift relation such as

$$E(a + 1) \circ P_{a+}(a) = Q_{a+}(a) \circ E(a) \quad (\textit{resp. } E(a - 1) \circ P_{a-}(a) = Q_{a-}(a) \circ E(a)),$$

where $Q_{a\pm}$ are some operators of the same order of $P_{a\pm}$.

In the following, we list the operators $P_{a\pm}$ and $Q_{a\pm}$ for each equation and each shift.

For the Gauss equation $E_2$, the following shift operators are classically known:

$$\begin{array}{lll lll} P_{a+} &=& x * dx + a, & Q_{a+} &=& x * dx + a + 1, \\ P_{a-} &=& x(x - 1) * dx + a + bx - c, & Q_{a-} &=& x(x - 1) * dx + a + bx - c + x - 1, \\ P_{c+} &=& (x - 1) * dx + a + b - c, & Q_{c+} &=& P_{c+}, \\ P_{c-} &=& x * dx + c - 1, & Q_{c-} &=& P_{c-}. \end{array}$$

For the generalized hypergeometric equations ${}_4E_3$ and ${}_3E_2$, they are fully described in the paper §14.

For the equations $E_4$ and $SE_4$, we find only a simple shift operator, which are mentioned in the paper. For the equation $E_3$, we could not find a shift operator.

For other equations, refer to the list of shift operators given in

| | |
|---|---|
| **E6PQ.txt** | for E6, |
| **SE6PQ.txt** | for SE6, |
| **E5PQ.txt** | for E5, |
| **SE5PQ.txt** | for SE5, |
| **Z4PQ.txt** | for Z4, |
| **ST4PQ.txt** | for ST4, |
| **Z3PQ.txt** | for Z3, |
| **SE3.txt** | for SE3, |
| **E3aPQ.txt** | for E3a, |
| **E3bPQ.txt** | for E3b, |
| **E3cPQ.txt** | for E3c, |
| **E3dPQ.txt** | for E3d. |

The equations $E_{3a}, \ldots, E_{3d}$ that are not in the figure above are specializations of $E_3$ treated in §12.5, which admits shift operators.