Asir

by Masayuki Noro, Takeshi Shimoyama, Taku Takeshima and Risa/Asir committers

1 Introduction

1.1 Organization of the Manual

This manual is organized as follows.

1. **Introduction**
   Organization of the Manual, notation and how to get Risa/Asir

2. **Risa/Asir**
   Summary of **Asir**, Installation

3. **Types**
   Types in **Asir**

4. **Asir user language**
   Description of **Asir** user language

5. **Debugger**
   Description of the debugger of **Asir** user language

6. **Built-in function**
   Detailed description of various built-in functions

7. **Distributed computation**
   Description of functions for distributed computation

8. **Groebner bases**
   Description of functions and operations for Groebner basis computation

9. **Algebraic numbers**
   Description of functions and operations for algebraic numbers

10. **Finite fields**
    Description of functions and operations on finite fields

11. **Appendix**
    Syntax in detail, description of sample files, interfaces for input from keyboard, changes, references

1.2 Notation

In this manual, we shall use several notations, which is described here

- The name of a function is written in a `typewriter` type
  `gcd()`, `gr()`

- For the description of a function, its argument is written in a *slanted* type.
  int, *poly*

- A file name is written in a '`typewriter type with single quotes`'
  '`dbxinit`', '`asir_plot`'

- An example is indented and written in a `typewriter` type.

      [0] 1;
      1
      [1] quit;

- References are made by a `typewriter type bracketed by []`.
  `[Boehm,Weiser]`
- Arguments (actual parameters) of a function are optional when they are bracketed by `[]`'s. The repeatable items (including non-existence of the item) are bracketed by `[]`*'s.
  `setprec([`*n*`]), diff(`*rat*`[,`*varn*`]*)`
- The prompt from the shell (csh) is denoted, as it is, by `%`. The prompt, however, is denoted by `#`, when you are assumed to be working as the root, for example, at the installation.

```
% cat afo
afo
bfo
% su
Password:XXXX
# cp asir /usr/local/bin
# exit
%
```

- The rational integer ring is denoted by **Z**, the rational number field by **Q**, the real number field by **R**, and the complex number field by **C**.

1.3 How to get Risa/Asir

Risa/Asir is available via http from

   `http://www.math.kobe-u.ac.jp/Asir`

1.4 Asir-contrib packages

Asir-contrib is a collection of packages developed by user communities. Documents can be found at `http://www.math.kobe-u.ac.jp/OpenXM/Current/doc/index-doc.html` This page also contains links to a manual on experimental functions of Risa/Asir and documents on the internal structure of Risa/Asir. As for bugs on Asir-contrib, please contact directly to the developer of the package. If you do not know who developes the package or there is no maintener on the package now, please send a mail to asir2005@math.kobe-u.ac.jp.

2 Risa/Asir

2.1 Risa and Asir

Risa is the name of whole libraries of a computer algebra system which is under development at FUJITSU LABORATORIES LIMITED. The structure of **Risa** is as follows.

- **The basic algebraic engine**
  This is the part which performs basic algebraic operations, such as arithmetic operations, to algebraic objects, e.g., numbers and polynomials, which are already converted into internal forms. It exists, like 'libc.a' of UNIX, as a library of ordinary UNIX system. The algebraic engine is written mainly in C language and partly in assembler. It serves as the basic operation part of **Asir**, a standard language interface of **Risa**.

- **Memory Manager**
  Risa employs, as its memory management component (the memory manager), a free software distributed by Boehm (`gc-6.1alpha5`). It is proposed by [Boehm,Weiser], and developed by Boehm and his colleagues. The memory manager has a memory allocator which automatically reclaims garbages, i.e., allocated but unused memories, and refreshes them for further use. The algebraic engine gets all its necessary memories through the memory manager.

- **Asir**
  Asir is a standard language interface of **Risa**'s algebraic engine. It is one of the possible language interfaces, because one can develop one's own language interface easily on **Risa** system. **Asir** is an example of such language interfaces. **Asir** has very similar syntax and semantics as C language. Furthermore, it has a debugger that provide a subset of commands of `dbx`, a widely used debugger of C language.

2.2 Features of Asir

As mentioned in the previous section, **Asir** is a standard language interface for**Risa**'s algebraic engine. Usually, it is provided as an executable file named `asir`. Main features supported for the current version of Asir is as follows.

- A C-like programming language
- Arithmetic operations (addition, subtraction, multiplication and division) on numbers, polynomials and rational expressions
- Operations on vectors and matrices
- List processing operations at the minimum
- Several Built-in functions (factorization, GCD computation, Groebner basis computation etc.)
- Useful user defined functions(e.g., factorization over algebraic number fields)
- A `dbx`-like debugger
- Plotting of implicit functions
- Numerical evaluation of mathematical expressions including elementary transcendental functions at arbitrary precision. This feature is in force only if **PARI** system (see Section 6.1.14 [pari], page 41).
- Distributed computation over UNIX

2.3 Installation

Any questions and any comments on this manual are welcome by e-mails to the following address.

```
noro@math.kobe-u.ac.jp
```

2.3.1 UNIX binary version

A file '`asir.tgz`' suitable for the target machine/architecture is required. After getting it, you have to unpack it by `gzip`. First of all, determine a directory where binaries and library files are installed. We call the directory the **library directory**. The following installs the files in '`/usr/local/lib/asir`'.

```
# gzip -dc asir.tgz | ( cd /usr/local/lib; tar xf - )
```

In this case you don't have to set any environment variable.

You can install them elsewhere.

```
% gzip -dc asir.tgz | ( cd $HOME; tar xf - )
```

In this case you have to set the name of the library directory to the environment variable `ASIR_LIBDIR`.

```
% setenv ASIR_LIBDIR $HOME/asir
```

Asir itself is in the library directory. It will be convenient to create a symbolic link to it from '`/usr/local/bin`' or the user's search path.

```
# ln -s /usr/local/lib/asir/asir /usr/local/bin/asir
```

Then you can start '`asir`'.

```
% /usr/local/bin/asir
This is Risa/Asir, Version 20000821.
Copyright (C) FUJITSU LABORATORIES LIMITED.
1994-2000. All rights reserved.
[0]
```

2.3.2 UNIX source code version

First of all you have to determine the install directory. In the install directory, the following subdirectories are put:

- bin

  executables of PARI and Asir
- lib

  library files of PARI and Asir
- include

  header files of PARI

These subdirectories are created automatically if they does not exist. If you can be a root, it is recommended to set the install directory to '`/usr/local`'. In the following the directory is denoted by TARGETDIR.

Then, install PARI library. After getting '`pari.tgz`', unpack and install it as follows:

```
% gzip -dc pari.tgz | tar xvf -
```

```
% cd pari
% ./Configure --prefix=TARGETDIR
% make all
% su
# make install
# make install-lib-sta
```

While executing 'make install', the procedure may stop due to some error. Then try the following:

```
% cd Oxxx
% make lib-sta
% su
# make install-lib-sta
# make install-include
# exit
%
```

In the above example, xxx denotes the name of the target operating system. Although GP is not built, the library necessary for building asir2000 will be generated.

After getting 'asir2000.tgz', unpack it and install necessary files as follows.

```
% gzip -dc asir.tgz | tar xf -
% cd asir2000
% ./configure --prefix=TARGETDIR --with-pari --enable-plot
% make
% su
# make install
# make install-lib
# make install-doc
# exit
```

2.3.3 Windows version

The installers are 'asirwin32.msi' (32bit version) and 'asirwin64.msi' (64bit version). These installers set the installed folder to the environmental variable 'ASIR_ROOTDIR'. (If you install Risa/Asir by the zip version, you need to set the environmental variable 'ASIR_ROOTDIR' by hand. You also need to unlock the security block before unzipping. The zip version is for developers.) Double click the "asirgui" on the desktop, then the asir starts. To use a unified environment with text editors, follow the instruction in '%ASIR_ROOTDIR%\share\editor'. If the folder name to which you install asir or your home folder name contain non-ascii characters, some functions of asir may not work properly. For example, in the Japense locale, names which may cause troubles can be checked by "damemoji checker".

ChangeLog

- Signals are emulated by threads. Some bugs related to signals are fixed by the following commit (2017-08-30).
- 1.13 gc-7.0-risa.diff, 1.37 parif.c, 1.101 ca.h, 1.40 ox.c, 1.79 ox_asir.c, 1.17 gc_risa.c, 1.97 glob.c, 1.30 ox_plot.c, 1.9 io.c, 1.12 plot.c
- Windows version has had a bug in bsave and bload. It is fixed by the following changes.

- 1.18 bload.c, 1.19 bsave.c, 1.22 io.c

2.4 Command line options

Command-line options for the command 'asir' are as follows.

-heap *number*

> In **Risa/Asir**, 4KB is used as an unit, called block, for memory allocation. By default, 16 blocks (64KB) are allocated initially. This value can be changed by giving an option **-heap** a number parameter in unit block. Size of the heap area is obtained by a Built-in function heap(), the result of which is a number in Bytes.

-adj *number*

> Heap area will be stretched by the memory manager, if the size of reclaimed memories is less than 1/*number* of currently allocated heap area. The default value for *number* is 3. If you do not prefer to stretch heap area by some reason, perhaps by restriction of available memories, but if prefer to resort to reclaiming garbages as far as possible, then a large value should be chosen for *number*, e.g., 8.

-norc When this option is specified, **Asir** does not read the initial file '`$HOME/.asirrc`'.

-quiet

-f *file* Instead of the standard input, *file* is used as the input. Upon an error, the execution immediately terminates.

-paristack *number*

> This option specifies the private memory size for PARI (see Section 6.1.14 [pari], page 41). The unit is Bytes. By default, it is set to 1 MB.

-maxheap *number*

> This option sets an upper limit of the heap size. The unit is Bytes. Note that the size is already limited by the value of `datasize` displayed by the command `limit` on UNIX.

-nofep Turn off the line edit functionality.

2.5 Environment variable

There exist several environment variables concerning with an execution of **Asir**. On UNIX, an environment variable is set from shells, or in rc files of shells. On Windows, it can be set from [Editing environmental variables].

- `ASIR_LIBDIR`
  This environmental variable is obsolete.
- `ASIR_CONTRIB_DIR`
  This environmental variable is obsolete.
- `ASIRLOADPATH`
  This environment specifies directories which contains files to be loaded by **Asir** command `load()`. Directories are separated by a ':' on UNIX, a ';' on

Windows respectively. The search order is from the left to the right. After searching out all directories in `ASIRLOADPATH`, or in case of no specification at all, the library directory will be searched. The library directories of the unix version are '`$OpenXM_HOME/lib/asir-contrib`', '`$OpenXM_HOME/lib/asir`', /usr/local/lib/asir-contrib, /usr/local/lib/asir. When '`OpenXM_HOME`' is not set, the library directories of the Windows version are %ASIR_ROOTDIR%\lib\asir-contrib, %ASIR_ROOTDIR%\lib\asir. When '`OpenXM_HOME`' is set, the library directories are same with the unix version. In the Windows version, the private folder '`%APPDATA%\OpenXM\lib\asir-contrib`' is also set to the library folders. In the unix version, there is no default private library folder. In the OpenXM/unix version, '`$OpenXM_tmp/OpenXM/lib/asir-contrib`' is added to the '`ASIRLOADPATH`' by a script in '`OpenXM/rc/`'. See also asir_contrib_update. See which, ctrl("loadpath"), asir2000/parse/load.c.

- `HOME`
  If **Asir** is invoked without *-norc*, '`$HOME/.asirrc`', if exists, is executed. If `HOME` is not set, nothing is done on UNIX. On Windows, '`.asirrc`' in **Asir root directory** is executed if it exists.

2.6 Starting and Terminating an Asir session

Run **Asir**, then the copyright notice and the first prompt will appear on your screen, and a new **Asir** session will be started.

```
[0]
```

When initialization file '`$HOME/.asirrc`' exists, **Asir** interpreter executes it at first taking it as a program file written in **Asir**.

The prompt indicates the sequential number of your input commands to **Asir**. The session will terminate when you input `end;` or `quit;` to **Asir**. Input commands are evaluated statement by statement. A statement normally ends with its terminator ';' or '$'. (There are some exceptions. See, syntax of **Asir**.) The result will be displayed when the command, i.e. statement, is terminated by a ';', and will not when terminated by a '$'. Note that after asir version 20240903, consecutive semicolons ';;' is considered as '$'. Therefore, for example, you cannot write '`for (;;)`'. You must write '`for (; ;)`' (add a space between consecutive semicolons).

```
% asir
[0] A;
0
[1] A=(x+y)^5;
x^5+5*y*x^4+10*y^2*x^3+10*y^3*x^2+5*y^4*x+y^5
[2] A;
x^5+5*y*x^4+10*y^2*x^3+10*y^3*x^2+5*y^4*x+y^5
[3] a=(x+y)^5;
evalpv : invalid assignment
return to toplevel
[3] a;
a
[4] fctr(A);
```

```
[[1,1],[x+y,5]]
[5] quit;
%
```

In the above example, names `A`, `a`, `x` and `y` are used to identify mathematical and programming objects. There, the name `A` denotes a program variable (some times called simply as a program variable.) while the other names, `a`, `x` and `y`, denote mathematical objects, that is, indeterminates. In general, program variables have names which begin with capital letters, while names of indeterminates begin with small letters. As you can see in the example, program variables are used to hold and keep objects, such as numbers and expressions, as their values, just like variables in C programming language. Whereas, indeterminates cannot have values so that assignment to indeterminates are illegal. If one wants to get a result by substituting a value for an indeterminate in an expression, it is achieved by the function `subst` as the value of the function.

2.7 Interruption

To interrupt the **Asir** execution, input an interrupt character from the keyboard. A `C-c` is usually used for it. (Notice: `C-x` on Windows and DOS.)

```
@ (x+y)^1000;
C-cinterrupt ?(q/t/c/d/u/w/?)
```

Here, the meaning of options are as follows.

q   Terminates **Asir** session. (Confirmation requested.)

t   Returns to toplevel. (Confirmation requested.)

c   Resumes to continue the execution.

d   Enters debugging mode at the next statement of the **Asir** program, if **Asir** has been executing a program loaded from a file. Note that it will sometimes take a long time before entering debugging mode when **Asir** is executing basic functions in the algebraic engine, (e.g., arithmetic operation, factorization etc.) Detailed description about the debugger will be given in Chapter 5 [Debugger], page 31.

u   After executing a function registered by `register_handler()` (see Section 7.5.6 [ox_reset ox_intr register_handler], page 110), returns to toplevel. A confirmation is prompted.

w   Displays the calling sequence up to the interruption.

?   Show a brief description of options.

2.8 Error handling

When arguments with illegal types are given to a built-in function, an error will be detected and the execution will be quit. In many cases, when an error is detected in a built-in function, **Asir** automatically enters debugging mode before coming back to toplevel. At that time, one can examine the state of the program, for example, inspect argument values just before the error occurred. Messages reported there are various depending on cases.

They are reported after the internal function name. The internal function name sometimes differs from the built-in function name that is specified by the user program.

In the execution of internal functions, errors may happen by various reasons. The UNIX version of **Asir** will report those errors as one of the following `internal error`'s, and enters debugging mode just like normal errors.

`SEGV`
`BUS ERROR`

> Some of the built-in functions transmit their arguments to internal operation routines without strict type-checking. In such cases, one of these two errors will be reported when an access violation caused by an illegal pointer or a NULL pointer is detected.

`BROKEN PIPE`

> In the process communication, this error will be reported if a process attempts to read from or to write onto the partner process when the stream to the partner process does not already exist, (e.g., terminated process.)

For UNIX version, even in such a case, the process itself does not terminate because such an error can be caught by `signal()` and recovered. To remove this weak point, complete type checking of all arguments are indispensable at the entry of a built-in function, which requires an enormous amount of re-making efforts.

2.9 Referencing results and special numbers

An `@` used for an escape character; rules currently in force are as follows.

`@n`   The evaluated result of n-th input command

`@@`   The evaluated result of the last command

`@i`   The unit of imaginary number, square root of -1.

`@pi`   The number pi, the ratio of a circumference of the circle and its diameter.

`@e`   Napier's number, the base of natural logarithm.

`@`   A generator of GF(2^m), a finite field of characteristic 2, over GF(2). It is a root of an irreducible univariate polynomial over GF(2) which is set as the defining polynomial of GF(2^m).

`@>, @<, @>=, @<=, @==, @&&, @||`
        Fist order logical operators. They are used in quantifier elimination.

```
[0] fctr(x^10-1);
[[1,1],[x-1,1],[x+1,1],[x^4+x^3+x^2+x+1,1],[x^4-x^3+x^2-x+1,1]]
[1] @@[3];
[x^4+x^3+x^2+x+1,1]
[2] eval(sin(@pi/2));
1.000000000000000000000000000000000000000000000000000000000
[3] eval(log(@e),20);
0.99999999999999999999999999998
```

```
[4] @0[4][0];
x^4-x^3+x^2-x+1
[5] (1+@i)^5;
(-4-4*@i)
[6] eval(exp(@pi*@i));
-1.0000000000000000000000000000
[7] (@+1)^9;
(@^9+@^8+@+1)
```

As you can see in the above example, results of toplevel computation can be referred to by
@ convention. This is convenient for users, while it sometimes imposes a heavy burden to
the garbage collector. It may happen that GC time will rapidly increase after computing a
very large expression at the toplevel. In such cases `delete_history()` (see Section 6.14.15
[`delete_history`], page 100) takes effect.

3 Data types

3.1 Types in Asir

In **Asir**, various objects described according to the syntax of **Asir** are translated to intermediate forms and by **Asir** interpreter further translated into internal forms with the help of basic algebraic engine. Such an object in an internal form has one of the following types listed below. In the list, the number coincides with the value returned by the built-in function `type()`. Each example shows possible forms of inputs for **Asir**'s prompt.

0 **0**

> As a matter of fact, no object exists that has 0 as its identification number. The number 0 is implemented as a null (0) pointer of C language. For convenience's sake, a 0 is returned for the input `type(0)`.

1 **number**

> ```
> 1 2/3  14.5  3+2*@i
> ```
>
> Numbers have sub-types. See Section 3.2 [Types of numbers], page 14.

2 **polynomial** (but not a number)

> ```
> x  afo  (2.3*x+y)^10
> ```
>
> Every polynomial is maintained internally in its full expanded form, represented as a nested univariate polynomial, according to the current variable ordering, arranged by the descending order of exponents. (See Section 8.1 [Distributed polynomial], page 121.) In the representation, the indeterminate (or variable), appearing in the polynomial, with maximum ordering is called the **main variable**. Moreover, we call the coefficient of the maximum degree term of the polynomial with respect to the main variable the **leading coefficient**.

3 **rational expression** (not a polynomial)

> ```
> (x+1)/(y^2-y-x)  x/x
> ```
>
> Note that in **Risa/Asir** a rational expression is not simplified by reducing the common divisors unless `red()` is called explicitly, even if it is possible. This is because the GCD computation of polynomials is a considerably heavy operation. You have to be careful enough in operating rational expressions.

4 **list**

> ```
> []  [1,2,[3,4],[x,y]]
> ```
>
> Lists are all read-only object. A null list is specified by `[]`. There are operations for lists: `car()`, `cdr()`, `cons()` etc. And further more, element referencing by indexing is available. Indexing is done by putting `[index]`'s after a program variable as many as are required. For example,
>
> ```
> [0] L = [[1,2,3],[4,[5,6]],7]$
> [1] L[1][1];
> [5,6]
> ```
>
> Notice that for lists, matrices and vectors, the index begins with number 0. Also notice that referencing list elements is done by following pointers from

the first element. Therefore, it sometimes takes much more time to perform referencing operations on a large list than on a vectors or a matrices with the same size.

5 vector

```
newvect(3)  newvect(2,[a,1])
```

Vector objects are created only by explicit execution of `newvect()` command. The first example above creates a null vector object with 3 elements. The other example creates a vector object with 2 elements which is initialized such that its 0-th element is `a` and 1st element is `1`. The second argument for `newvect` is used to initialize elements of the newly created vector. A list with size smaller or equal to the first argument will be accepted. Elements of the initializing list is used from the left to the right. If the list is too short to specify all the vector elements, the unspecified elements are filled with as many 0's as are required. Any vector element is designated by indexing, e.g., `[index]`. `Asir` allows any type, including vector, matrix and list, for each respective element of a vector. As a matter of course, arrays with arbitrary dimensions can be represented by vectors, because each element of a vector can be a vector or matrix itself. An element designator of a vector can be a left value of assignment statement. This implies that an element designator is treated just like a simple program variable. Note that an assignment to the element designator of a vector has effect on the whole value of that vector.

```
[0] A3 = newvect(3);
[ 0 0 0 ]
[1] for (I=0;I<3;I++)A3[I] = newvect(3);
[2] for (I=0;I<3;I++)for(J=0;J<3;J++)A3[I][J]=newvect(3);
[3] A3;
[ [ [ 0 0 0 ] [ 0 0 0 ] [ 0 0 0 ] ]
[ [ 0 0 0 ] [ 0 0 0 ] [ 0 0 0 ] ]
[ [ 0 0 0 ] [ 0 0 0 ] [ 0 0 0 ] ] ]
[4] A3[0];
[ [ 0 0 0 ] [ 0 0 0 ] [ 0 0 0 ] ]
[5] A3[0][0];
[ 0 0 0 ]
```

6 matrix

```
newmat(2,2)  newmat(2,3,[[x,y],[z]])
```

Like vector objects, matrix objects are also created only by explicit execution of `newmat()` command. Initialization of the matrix elements are done in a similar manner with that of the vector elements except that the elements are specified by a list of lists. Each element, again a list, is used to initialize each row; if the list is too short to specify all the row elements, unspecified elements are filled with as many 0's as are required. Like vectors, any matrix element is designated by indexing, e.g., `[index][index]`. `Asir` also allows any type, including vector, matrix and list, for each respective element of a matrix. An element designator of a matrix can also be a left value of assignment statement. This implies that an element designator is treated just like a simple program variable. Note that

an assignment to the element designator of a matrix has effect on the whole value of that matrix. Note also that every row, (not column,) of a matrix can be extracted and referred to as a vector.

```
[0] M=newmat(2,3);
[ 0 0 0 ]
[ 0 0 0 ]
[1] M[1];
[ 0 0 0 ]
[2] type(@@);
5
```

7 string

```
""  "afo"
```

Strings are used mainly for naming files. It is also used for giving comments of the results. Operator symbol + denote the concatenation operation of two strings.

```
[0] "afo"+"take";
afotake
```

8 structure

```
newstruct(afo)
```

The type **structure** is a simplified version of that in C language. It is defined as a fixed length array and each entry of the array is accessed by its name. A name is associated with each structure.

9 distributed polynomial

```
2*<<0,1,2,3>>-3*<<1,2,3,4>>
```

This is the short for 'Distributed representation of polynomials.' This type is specially devised for computation of Groebner bases. Though for ordinary users this type may never be needed, it is provided as a distinguished type that user can operate by `Asir`. This is because the Groebner basis package provided with `Risa/Asir` is written in the `Asir` user language. For details See Chapter 8 [Groebner basis computation], page 121.

10 32bit unsigned integer
11 error object

These are special objects used for OpenXM.

12 matrix over GF(2)

This is used for basis conversion in finite fields of characteristic 2.

13 MATHCAP object

This object is used to express available functionalities for Open XM.

14 first order formula

This expresses a first order formula used in quantifier elimination.

15 **matrix over GF(p)**

  A matrix over a small finite field.

16 **byte array**

  An array of unsigned bytes.

26 **distributed module polynomial**
      2*<<0,1,2,3:1>>-3*<<1,2,3,4:2>>

  This represents an element in a free module over a polynomial ring as a lin-
  ear sum of module monomials, where a module monomial is the product of a
  monomial in the polynomial ring and a standard base of the free module. For
  details See Chapter 8 [Groebner basis computation], page 121.

-1 **VOID object**

  The object with the object identifier -1 indicates that a return value of a function
  is void.

3.2 Types of numbers

0   **rational number**
          Rational numbers are implemented by arbitrary precision integers (**bignum**). A
          rational number is always expressed by a fraction of lowest terms.

1   **double precision floating point number (double float)**
          The numbers of this type are numbers provided by the computer hardware. By
          default, when **Asir** is started, floating point numbers in a ordinary form are
          transformed into numbers of this type. However, they will be transformed into
          bigfloat numbers when the switch **bigfloat** is turned on (enabled) by `ctrl()`
          command.

              [0] 1.2;
              1.2
              [1] 1.2e-1000;
              0
              [2] ctrl("bigfloat",1);
              1
              [3] 1.2e-1000;
              1.2000000000000000513 E-1000

          A rational number shall be converted automatically into a double float number
          before the operation with another double float number and the result shall be
          computed as a double float number.

2   **algebraic number**
          See Chapter 9 [Algebraic numbers], page 162.

3   **bigfloat**
          The **bigfloat** numbers of **Asir** is realized by **MPFR** library. A **bigfloat** number
          of **MPFR** has an arbitrary precision mantissa part. However, its exponent part

admits only a 64bit integer. Floating point operations will be performed all in **bigfloat** after activating the **bigfloat** switch by `ctrl()` command. The default precision is 53 bits (about 15 digits), which can be specified by `setbprec()` and `setprec()` command.

```
[0] ctrl("bigfloat",1);
1
[1] eval(2^(1/2));
1.4142135623731
[2] setprec(100);
15
[3] eval(2^(1/2));
1.41421356237309504880168872420969807856967187537694...764157
[4] setbprec(100);
332
[5] 1.41421356237309504880168872421
```

Function `eval()` evaluates numerically its argument as far as possible. Notice that the integer given for the argument of `setbprec()` guarantees the accuracy of the result according to the current rounding mode. The argument of `setbprec()` is converted to the corresponding bit length and set. (See Section 6.1.13 [eval deval], page 40.)

4   **complex number**
         A **complex** number of **Risa/Asir** is a number with the form `a+b*@i`, where @i is the unit of imaginary number, and `a` and `b` are either a **rational** number, **double float** number or **bigfloat** number, respectively. The real part and the imaginary part of a **complex** number can be taken out by `real()` and `imag()` respectively.

5   **element of a small finite prime field**
         Here a small finite field means that its characteristic is less than 2^27. At present small finite fields are used mainly for groebner basis computation, and elements in such finite fields can be extracted by taking coefficients of distributed polynomials whose coefficients are in finite fields. Such an element itself does not have any information about the field to which the element belongs, and field operations are executed by using a prime p which is set by `setmod()`.

6   **element of large finite prime field**
         This type expresses an element of a finite prime field whose characteristic is an arbitrary prime. An object of this type is obtained by applying `simp_ff` to an integer.

7   **element of a finite field of characteristic 2**
         This type expresses an element of a finite field of characteristic 2. Let F be a finite field of characteristic 2. If [F:GF(2)] is equal to n, then F is expressed as F=GF(2)[t]/(f(t)), where f(t) is an irreducible polynomial over GF(2) of degree n. As an element g of GF(2)[t] can be expressed by a bit string, An element g $mod f$ in F can be expressed by two bit strings representing g and f respectively.

         Several methods to input an element of F are provided.

- @
  @ represents *t mod f* in F=GF(2)[t](f(t)). By using @ one can input an element of *F*. For example `@^10+@+1` represents an element of *F*.
- ptogf2n
  `ptogf2n` converts a univariate polynomial into an element of *F*.
- ntogf2n
  As a bit string, a non-negative integer can be regarded as an element of *F*. Note that one can input a non-negative integer in decimal, hexadecimal (`0x` prefix) and binary (`0b` prefix) formats.
- micellaneous
  `simp_ff` is available if one wants to convert the whole coefficients of a polynomial.

8   **element of a finite field of characteristic $p\hat{\ }n$**

A finite field of order $p\hat{\ }n$, where *p* is an arbitrary prime and *n* is a positive integer, is set by `setmod_ff` by specifying its characteristic *p* and an irreducible polynomial of degree *n* over GF(*p*). An element of this field is represented by a polynomial over GF(*p*) modulo m(x).

9   **element of a finite field of characteristic $p\hat{\ }n$ (small order)**

A finite field of order $p\hat{\ }n$, where $p\hat{\ }n$ must be less than $2\hat{\ }29$ and *n* must be equal to 1 if *p* is greater or equal to $2\hat{\ }14$, is set by `setmod_ff` by specifying its characteristic *p* the extension degree *n*. If *p* is less than $2\hat{\ }14$, each non-zero element of this field is a power of a fixed element, which is a generator of the multiplicative group of the field, and it is represented by its exponent. Otherwise, each element is represented by the redue modulo *p*. This specification is useful for treating both cases in a single program.

10   **element of a finite field which is an algebraic extension of a small finite field of characteristic $p\hat{\ }n$**

An extension field *K* of the small finite field *F* of order $p\hat{\ }n$ is set by `setmod_ff` by specifying its characteristic *p* the extension degree *n* and m=[K:F]. An irreducible polynomial of degree *m* over *K* is automatically generated and used as the defining polynomial of the generator of the extension *K/F*. The generator is denoted by `@s`.

11   **algebraic number represented by a distributed polynomial**
     See Chapter 9 [Algebraic numbers], page 162.

   Finite fields other than small finite prime fields are set by `setmod_ff`. Elements of finite fields do not have information about the modulus. Upon an arithmetic operation, if one of the operands is a rational number, it is automatically converted into an element of the finite field currently set and the operation is done in the finite field.

3.3 Types of indeterminates

An algebraic object is recognized as an indeterminate when it can be a (so-called) variable in polynomials. An ordinary indeterminate is usually denoted by a string that start with a small alphabetical letter followed by an arbitrary number of alphabetical letters, digits or

'_'. In addition to such ordinary indeterminates, there are other kinds of indeterminates in a wider sense in **Asir**. Such indeterminates in the wider sense have type **polynomial**, and further are classified into sub-types of the type **indeterminate**.

0   **ordinary indeterminate**
           An object of this sub-type is denoted by a string that start with a small al-
           phabetical letter followed by an arbitrary number of alphabetical letters, digits
           or '_'. This kind of indeterminates are most commonly used for variables of
           polynomials.

```
[0] [vtype(a),vtype(aA_12)];
[0,0]
```

1   **undetermined coefficient**
           The function `uc()` creates an indeterminate which is denoted by a string that
           begins with '_'. Such an indeterminate cannot be directly input by its name.
           Other properties are the same as those of **ordinary indeterminate**. Therefore,
           it has a property that it cannot cause collision with the name of ordinary
           indeterminates input by the user. And this property is conveniently used to
           create undetermined coefficients dynamically by programs.

```
[1] U=uc();
_0
[2] vtype(U);
1
```

2   **function form**
           A function call to a built-in function or to an user defined function is usually
           evaluated by **Asir** and retained in a proper internal form. Some expressions,
           however, will remain in the same form after evaluation. For example, `sin(x)`
           and `cos(x+1)` will remain as if they were not evaluated. These (unevaluated)
           forms are called 'function forms' and are treated as if they are indeterminates
           in a wider sense. Also, special forms such as `@pi` the ratio of circumference and
           diameter, and `@e` Napier's number, will be treated as 'function forms.'

```
[3] V=sin(x);
sin(x)
[4] vtype(V);
2
[5] vars(V^2+V+1);
[sin(x)]
```

3   **functor**
           A function call (or a function form) has a form *fname*(*args*). Here, *fname* alone
           is called a **functor**. There are several kinds of **functor**s: built-in functor, user
           defined functor and functor for the elementary functions. A functor alone is
           treated as an indeterminate in a wider sense.

```
[6] vtype(sin);
3
```

4 User language Asir

Asir provides many built-in functions, which perform algebraic computations, e.g., factorization and GCD computation, file I/O, extract a part of an algebraic expression, etc. In practice, you will often encounter a specific problem for which **Asir** does not provide a direct solution. For such cases, you have to write a program in a certain user language. The user language for **Asir** is also called **Asir**. In the following, we describe the Syntax and then show how to write a user program by several examples.

4.1 Syntax — Difference from C language

The syntax of **Asir** is based on C language. Main differences are as follows. In this section, a variable does not mean an indeterminate, but a program variable which is written by a string which begins with a capital alphabetical letter in **Asir**.

- No types for variables.
  As is already mentioned, any object in **Asir** has their respective types. A program variable, however, is type-less, that is, any typed object can be assigned to it.

  ```
  [0] A = 1;
  1
  [1] type(A);
  1
  [2] A = [1,2,3];
  [1,2,3]
  [3] type(A);
  4
  ```

- Variables, together with formal parameters, in a function (procedure) are all local to the function by default.
  Variables can be global at the top level, if they are declared with the key word `extern`. Thus, the scope rule of **Asir** is very simple. There are only two types of variables: global variables and local variables. A name that is input to the **Asir**'s prompt at the top level is denotes a global variable commonly accessed at the top level. In a function (procedure) the following rules are applied.

  1. If a variable is declared as global by an `extern` statement in a function, the variable used in that function denotes a global variable at the top level. Furthermore, if a variable in a function is preceded by an `extern` declaration outside the function but in a file where the function is defined, all the appearance of that variable in the same file denote commonly a global variable at the top level.

  2. A variable in a function is local to that function, if it is not declared as global by an `extern` declaration.

     ```
     % cat afo
     def afo() { return A;}
     extern A$
     def bfo() { return A;}
     end$
     % asir
     [0] load("afo")$
     ```

```
[5] A = 1;
1
[6] afo();
0
[7] bfo();
1
```

- Program variables and algebraic indeterminates are distinguished in **Asir**.
  The names of program variables must begin with a capital letter; while the names of
  indeterminates and functions must begin with a small letter.

  This is an unique point that differs from almost all other existing computer algebra
  systems. The distinction between program variables and indeterminates is adopted to
  avoid the possible and usual confusion that may arise in a situation where a name is
  used as an indeterminate but, as it was, the name has been already assigned some
  value. To use different type of letters, capital and small, was a matter of syntactical
  convention like Prolog, but it is convenient to distinguish variables and indeterminates
  in a program.
- No `switch` statements, and `goto` statements.
  Lack of `goto` statement makes it rather bothering to exit from within multiple loops.
- Comma expressions are allowed only in `A`, `B` and `C` of the constructs `for (A;B;C)` or
  `while(A)`.
  This limitation came from adopting lists as legal data objects for **Asir**.

The above are limitations; extensions are listed as follows.

- Arithmetic for rational expressions can be done in the same manner as is done for
  numbers in C language.
- Lists are available for data objects.

  Lists are conveniently used to represent a certain collection of objects. Use of lists
  enables to write programs more easily, shorter and more comprehensible than use of
  structure like C programs.
- Options can be specified in calling user defined functions.

  See Section 4.2.12 [option], page 27.

4.2 Writing user defined functions

4.2.1 User defined functions

To define functions by an user himself, '`def`' statement must be used. Syntactical errors are
detected in the parsing phase of **Asir**, and notified with an indication of where **Asir** found
the error. If a function with the same name is already defined (regardless to its arity,) the
new definition will override the old one, and the user will be told by a message,

```
afo() redefined.
```

on the screen when a flag `verbose` is set to a non-zero value by `ctrl()`. Recursive definition,
and of course, recursive use of functions are available. A call for an yet undefined function
in a function definition is not detected as an error. An error will be detected at execution
of the call of that yet undefined function.

```
/* X! */

def f(X) {
    if ( !X )
        return 1;
    else
        return X * f(X-1);
}



/* iCj ( 0 ≤ i ≤ N, 0 ≤ j ≤ i ) */

def c(N)
{
    A = newvect(N+1); A[0] = B = newvect(1); B[0] = 1;
    for ( K = 1; K <= N; K++ ) {
        A[K] = B = newvect(K+1); B[0] = B[K] = 1;
        for ( P = A[K-1], J = 1; J < K; J++ )
            B[J] = P[J-1]+P[J];
        }
    return A;
}



/* A + B */

def add(A,B)
"add two numbers."
{
    return A+B;
}
```

In the second example, `c(N)` returns a vector, say `A`, of length `N+1`. `A[I]` is a vector of length `I+1`, and each element is again a vector which contains $_I C_J$ as its elements.

References
        Section 6.14.4 [`help`], page 95.

In the following, the manner of writing **Asir** programs is exhibited for those who have no experience in writing C programs.

4.2.2 variables and indeterminates

variables (program variables)

        A program variable is a string that begins with a capital alphabetical letter followed by any numbers of alphabetical letters, digits and '`_`'.

        A program variable is thought of a box (a carrier) which can contain **Asir** objects of various types. The content is called the 'value' of that variable. When an expression in a program is to be evaluated, the variable appearing in

the expression is first replaced by its value and then the expression is evaluated to some value and stored in the memory. Thus, no program variable appears in objects in the internal form. All the program variables are initialized to the value 0.

```
[0] X^2+X+1;
1
[1] X=2;
2
[2] X^2+X+1;
7
```

indeterminates

An indeterminate is a string that begins with a small alphabetical letter followed by any numbers of alphabetical letters, digits and '_'.

An indeterminate is a transcendental element, so-called variable, which is used to construct polynomial rings. An indeterminate cannot have any value. No assignment is allowed to it.

```
[3] X=x;
x
[4] X^2+X+1;
x^2+x+1
[5] A='Dx'*(x-1)+x*y-y;
(y+Dx)*x-y-Dx
[6] function foo(x,y);
[7] B=foo(x,y)*x^2-1;
foo(x,y)*x^2-1
```

4.2.3 parameters and arguments

```
def sum(N) {
    for ( I = 1, S = 0; I <= N; I++ )
        S += I;
    return S;
}
```

This is an example definition of a function that sums up integers from 1 to `N`. The `N` in `sum(N)` is called the (formal) parameter of `sum(N)`. The example shows a function of the single argument. In general, any number of parameters can be specified by separating by commas (','). A (formal) parameter accepts a value given as an argument (or an actual parameter) at a function call of the function. Since the value of the argument is given to the formal parameter, any modification to the parameter does not usually affect the argument (or actual parameter). However, there are a few exceptions: vector arguments and matrix arguments.

Let `A` be a program variable and assigned to a vector value `[ a, b ]`. If A is given as an actual parameter to a formal parameter, say `V`, of a function, then an assignment in the function to the vector element designator `V[1]`, say `V[1]=c;`, causes modification of the actual parameter `A` resulting `A` to have an altered value `[ a c ]`. Thus, if a vector is given to a formal parameter of a function, then its element (and subsequently the vector itself)

in the calling side is modified through modification of the formal parameter by a vector element designator in the called function. The same applies to a matrix argument. Note that, even in such case where a vector (or a matrix) is given to a formal parameter, the assignment to the whole parameter itself has only a local effect within the function.

```
def clear_vector(M) {
    /* M is expected to be a vector */
    L = size(M)[0];
    for ( I = 0; I < L; I++ )
        M[I] = 0;
}
```

This function will clear off the vector given as its argument to the formal parameter `M` and return a 0 vector.

Passing a vector as an argument to a function enables returning multiple results by packing each result in a vector element. Another alternative to return multiple results is to use a list. Which to use depends on cases.

4.2.4 comments

The text enclosed by '`/*`' and '`*/`' (containing '`/*`' and '`*/`') is treated as a comment and has no effect to the program execution as in C programs.

```
/*
 * This is a comment.
 */

def afo(X) {
```

A comment can span to several lines, but it cannot be nested. Only the first '`/*`' is effective no matter how many '`/*`''s in the subsequent text exist, and the comment terminates at the first '`*/`'.

In order to comment out a program part that may contain comments in it, use the pair, `#if 0` and `#endif`. (See Section 4.2.11 [preprocessor], page 26.)

```
#if 0
def bfo(X) {
/* empty */
}
#endif
```

4.2.5 statements

An user function of **Asir** is defined in the following form.

```
def name(parameter, parameter,...,parameter) {
    statement
    statement
    ...
    statement
}
```

As you can see, the statement is a fundamental element of the function. Therefore, in order to write a program, you have to learn what the statement is. The simplest statement is the simple statement. One example is an expression with a terminator (';' or '$'.)

```
S = sum(N);
```

A 'return statement' and 'break statement' are also primitives to construct 'statements.' As you can see the syntactic definition of 'if statement' and 'for statement', each of their bodies consists of a single 'statement.' Usually, you need several statements in such a body. To solve this contradictory requirement, you may use the 'compound statement.' A 'compound statement' is a sequence of 'statement's enclosed by a left brace '{' and a right brace '}'. Thus, you can use multiple statement as if it were a single statement.

```
if ( I == 0 ) {
    J = 1;
    K = 2;
    L = 3;
}
```

No terminator symbol is necessary after '}', because '{' statement sequence '}' already forms a statement, and it satisfies the syntactical requirement of the 'if statement.'

4.2.6 return statement

There are two forms of return statement.

```
return expression;
```

```
return;
```

Both forms are used for exiting from a function. The former returns the value of the expression as a function value. The function value of the latter is not defined.

4.2.7 if statement

There are two forms of if statement.

```
if ( expression )                  if ( expression )
    statement        and               statement
else
    statement
```

The interpretation of these forms are obvious. However, be careful when another if statement comes at the place for 'statement'. Let us examine the following example.

```
if ( expression1 )
    if ( expression2 ) statement1
else
    statement2
```

One might guess statement2 after else corresponds with the first if (expression1) by its appearance of indentation. But, as a matter of fact, the Asir parser decides that it correspond with the second if (expression2). Ambiguity due to such two kinds of forms of if statement is thus solved by introducing a rule that a statement preceded by an else matches to the nearest preceding if.

Therefore, rearrangement of the above example for improving readability according to the actual interpretation gives the following.

```
if ( expression1 ) {
    if ( expression2 ) statement1 else statement2
}
```

On the other hand, in order to reflect the indentation, it must be written as the following.

```
if ( expression1 ) {
    if ( expression2 ) statement1
} else
    statement2
```

When `if` is used in the top level, the `if` expression should be terminated with `$` or `;`. If there is no terminator, the next expression will be skipped to be evaluated.

4.2.8 loop, break, return, continue

There are three kinds of statements for loops (repetitions): the `while` statement, the `for` statement, and the `do` statement.

- `while` statement
  It has the following form.

  ```
  while ( expression ) statement
  ```

  This statement specifies that `statement` is repeatedly evaluated as far as the `expression` evaluates to a non-zero value. If the expression 1 is given to the `expression`, it forms an infinite loop.

- `for` statement
  It has the following form.

  ```
  for ( expr list-1; expr; expr list-2 ) statement
  ```

  This is equivalent to the program

  ```
  expr list-1 (transformed into a sequence of simple statement)
  while ( expr ) {
      statement
      expr list-2 (transformed into a sequence of simple statement)
  }
  ```

- do statement

  ```
  do {
      statement
  } while ( expression )
  ```

  This statement differs from `while` statement by the location of the termination condition: This statement first execute the `statement` and then check the condition, whereas `while` statement does it in the reverse order.

As means for exiting from loops, there are `break` statement and `return` statement. The `continue` statement allows to move the control to a certain point of the loop.

- `break`
  The `break` statement is used to exit the inner most loop.

- `return`
  The `return` statement is usually used to exit from a function call and it is also effective in a loop.

- `continue`
  The `continue` statement is used to move the control to the end point of the loop body. For example, the last expression list will be evaluated in a `for` statement, and the termination condition will be evaluated in a `while` statement.

4.2.9 structure definition

A structure data type is a fixed length array and each component of the array is accessed by its name. Each type of structure is distinguished by its name. A structure data type is declared by `struct` statement. A structure object is generated by a builtin function `newstruct`. Each member of a structure is accessed by an operator `->`. If a member of a structure is again a structure, then the specification by `->` can be nested.

```
[1] struct rat {num,denom};
0
[2] A = newstruct(rat);
{0,0}
[3] A->num = 1;
1
[4] A->den = 2;
2
[5] A;
{1,2}
[6] struct_type(A);
1
```

References
Section 6.7.1 [`newstruct`], page 73, Section 6.7.3 [`struct_type`], page 76

4.2.10 various expressions

Major elements to construct expressions are the following:

- addition, subtraction, multiplication, division, exponentiation
  The exponentiation is denoted by '`^`'. (This differs from C language.) Division denoted by '`/`' is used to operate in a field, for example, `2/3` results in a rational number `2/3`. For integer division and polynomial division, both including remainder operation, built-in functions are provided.

      x+1 A^2*B*afo X/3

- programming variables with indices
  An element of a vector, a matrix or a list can be referred to by indexing. Note that the indices begin with number 0. When the referred element is again a vector, a matrix or a list, repeated indexing is also effective.

      V[0] M[1][2]

- comparison operation
  There are comparison operations '`==`' for equivalence, '`!=`' for non-equivalence, '`>`',

'<','>=', and '<=' for larger or smaller. The results of these operations are either value 1 for the truth, or 0 for the false.

- logical expression
  There are two binary logical operations '&&' for logical 'conjunction'(and), '||' for logical 'disjunction'(or), and one unary logical operation '!' for logical 'negation'(not). The results of these operations are either value 1 for the truth, and 0 for the false.

- assignment
  Value assignment of a program variable is usually done by '='. There are special assignments combined with arithmetic operations. ('+=', '-=', '*=', '/=', '^=')

  ```
  A = 2  A *= 3 (the same as A = A*3; The others are alike.)
  ```

- function call
  A function call is also an expression.

- '++', '--'
  These operators are attached to or before a program variable, and denote special operations and values.

  ```
  A++  the expression value is the previous value of A, and A = A+1
  A--  the expression value is the previous value of A, and A = A-1
  ++A  A = A+1, and the value is the one after increment of A
  --A  A = A-1, and the value is the one after decrement of A
  ```

4.2.11 preprocessor

he **Asir** user language imitates C language. A typical features of C language include macro expansion and file inclusion by the preprocessor cpp. Also, **Asir** read in user program files through cpp. This enables **Asir** user to use #include, #define, #if etc. in his programs.

- #include
  Include files are searched within the same directory as the file containing #include so that no arguments are passed to cpp.

- #define
  This can be used just as in C language.

- #if
  This is conveniently used to comment out a large part of a user program that may contain comments by /* and */, because such comments cannot be nested.

the following are the macro definitions in 'defs.h'.

```
#define ZERO 0
#define NUM 1
#define POLY 2
#define RAT 3
#define LIST 4
#define VECT 5
#define MAT 6
#define STR 7
#define N_Q 0
#define N_R 1
#define N_A 2
```

```
#define N_B 3
#define N_C 4
#define V_IND 0
#define V_UC 1
#define V_PF 2
#define V_SR 3
#define isnum(a) (type(a)==NUM)
#define ispoly(a) (type(a)==POLY)
#define israt(a) (type(a)==RAT)
#define islist(a) (type(a)==LIST)
#define isvect(a) (type(a)==VECT)
#define ismat(a) (type(a)==MAT)
#define isstr(a) (type(a)==STR)
#define FIRST(L) (car(L))
#define SECOND(L) (car(cdr(L)))
#define THIRD(L) (car(cdr(cdr(L))))
#define FOURTH(L) (car(cdr(cdr(cdr(L)))))
#define DEG(a) deg(a,var(a))
#define LCOEF(a) coef(a,deg(a,var(a)))
#define LTERM(a) coef(a,deg(a,var(a)))*var(a)^deg(a,var(a))
#define TT(a) car(car(a))
#define TS(a) car(cdr(car(a)))
#define MAX(a,b) ((a)>(b)?(a):(b))
```

Since we are utilizing the C preprocessor, it cannot properly preprocess expressions with `$`. For example, even if `LIST` is defined, `LIST` in the expression `LIST$` is not replaced. Add a blank before `$`, i.e., write as `LIST $` to make the proprocessor replace it properly.

4.2.12 option

If a user defined function is declared with N arguments, then the function is callable with N arguments only.

```
[0] def factor(A) { return fctr(A); }
[1] factor(x^5-1,3);
evalf : argument mismatch in factor()
return to toplevel
```

A function with indefinite number of arguments can be realized by using a list or an array as its argument. Another method is available as follows:

```
% cat factor
def factor(F)
{
    Mod = getopt(mod);
    ModType = type(Mod);
    if ( ModType == 1 ) /* 'mod' is not specified. */
        return fctr(F);
    else if ( ModType == 0 ) /* 'mod' is a number */
        return modfctr(F,Mod);
}
```

```
[0] load("factor")$
[1] factor(x^5-1);
[[1,1],[x-1,1],[x^4+x^3+x^2+x+1,1]]
[2] factor(x^5-1|mod=11);
[[1,1],[x+6,1],[x+2,1],[x+10,1],[x+7,1],[x+8,1]]
```

In the second call of `factor()`, `|mod=11` is placed after the argument `x^5-1`, which appears in the declaration of `factor()`. This means that the value `11` is assigned to the keyword `mod` when the function is executed. The value can be retrieved by `getopt(mod)`. We call such machinery *option*. If the option for *mod* is not specified, `getopt(mod)` returns an object whose type is -1. By this feature, one can describe the behaviour of the function when the option is not specified by *if* statements. After '|' one can append any number of options separated by ',.'.

```
[100] xxx(1,2,x^2-1,[1,2,3]|proc=1,index=5);
```

Optional arguments may be given as a list with the key word `option_list` as `option_list=[["key1",value1],["key2",value2],...]`. It is equivalent to pass the optional arguments as `key1=value1,key2=value2,...`.

```
[101] dp_gr_main([x^2+y^2-1,x*y-1]|option_list=[["v",[x,y]],["order",[[x,5,y,1]]]]);
```

Since `getopt()` returns an option list, the optional argument `option_list=...` is useful when we call functions with optional arguments from a function with optional arguments to pass the all optional parameters.

```
% cat foo.rr
def foo(F)
{
    OPTS=getopt();
    return factor(F|option_list=OPTS);
}
[3] load("foo.rr")$
[4] foo(x^5-1|mod=11);
[[1,1],[x+6,1],[x+2,1],[x+10,1],[x+7,1],[x+8,1]]
```

4.2.13 module

Function names and variables in a library may be encapsulated by module. Let us see an example of using module

```
module stack;

static Sp $
Sp = 0$
static Ssize$
Ssize = 100$
static Stack $
Stack = newvect(Ssize)$
localf push $
localf pop $

def push(A) {
```

```
    if (Sp >= Ssize) {print("Warning: Stack overflow\nDiscard the top"); pop();}█
    Stack[Sp] = A;
    Sp++;
  }
  def pop() {
    local A;
    if (Sp <= 0) {print("Stack underflow"); return 0;}
    Sp--;
    A = Stack[Sp];
    return A;
  }
  endmodule;

  def demo() {
    stack.push(1);
    stack.push(2);
    print(stack.pop());
    print(stack.pop());
  }
```

Module is encapsulated by the sentences `module` module name and `endmodule`. A variable of a module is declared with the key word `static`. The static variables cannot be referred nor changed out of the module, but it can be referred and changed in any functions in the module. The `static` variables must be declared before the definitions of functions, because the one-path parser of asir automatically assume variables as local variables if there is no declaration for them. A global variable which can be referred and changed in or out of the module is declared with the key word `extern`.

Any function defined in a module must be declared forward with the keyword `localf`. In the example above, `push` and `pop` are declared. This declaration is necessary.

A function `functionName` defined in a module `moduleName` can be called by the expression `moduleName.functioName(arg1, arg2, ...)` out of the module. Inside the module, `moduleName.` is not necessary. In the example below, the functions `push` and `pop` defined in the module `stack` are called out of the module.

```
    stack.push(2);
    print( stack.pop() );
    2
```

Any function name defined in a module is local. In other words, the same function name may be used out of the module to define a different function.

The module structure of asir is introduced to develop large libraries. In order to load libraries on demand, the command `module_definedp` will be useful. The below is an example of demand loading.

```
    if (!module_definedp("stack")) load("stack.rr") $
```

It is not necessary to declare local variables in asir. As you see in the example of the stack module, we may declare local variables by the key word `local`. Once this key word is used, asir requires to declare all the variables. In order to avoid some troubles to develop a large libraries, it is recommended to use `local` declarations.

When we need to call a function in a module before the module is defined, we must make a prototype declaration as the example below.

```
/* Prototype declaration of the module stack */
module stack;
localf push $
localf pop $
endmodule;

def demo() {
  print("----------------");
  stack.push(1);
  print(stack.pop());
  print("--------------");
}


module stack;
  /* The body of the module stack */
endmodule;
```

In order to call functions defined in the top level from the inside of a module, we use `::` as in the example below.

```
def afo() {
  S = "afo, afo";
  return S;
}
module abc;
localf foo,afo $

def foo() {
  G = ::afo();
  return G;
}
def afo() {
  return "afo, afo in abc";
}
endmodule;
end$

[1200] abc.foo();
afo, afo
[1201] abc.afo();
afo, afo in abc
```

References
        Section 6.12.1 [module_list], page 89, Section 6.12.2 [module_definedp], page 89, Section 6.12.3 [remove_module], page 89.

5 Debugger

5.1 What is Debugger

A debugger `dbx` is available for C programs on `Sun`, `VAX` etc. In `dbx`, one can use commands such as setting break-point on a source line, stepwise execution, inspecting a variable's value etc. **Asir** provides such a `dbx`-like debugger. In addition to such commands, we adopted several useful commands from `gdb`. In order to enter the debug-mode, type `debug;` at the top level of **Asir**.

```
[10] debug;
(debug)
```

`Asir` also enters the debug-mode by the following means or in the following situations.

- When it reaches a break point while executing a program.
- When the 'd' option is selected at an interruption.
- When it detects errors while executing a program.

  In this case, to continue the execution of the program is impossible. But because it reports the statement in the user defined function that caused the error, then enters the debug-mode, user can inspect the values of variables at the error state. This helps to analyze the error and debug the program.

- When built-in function `error()` is called.

5.2 Debugger commands

Only indispensable commands of `dbx` are supported in the current version. Generally, the effect of a command is the same as that of `dbx`. There are, however, slight differences: Commands `step` and `next` execute the next statement, but not the next line; therefore, if there are multiple statements in one line, one should issue such commands several times to proceed the next line. The debugger reads in `.dbxinit`, which allows the same aliases as is used in `dbx`.

`step`	Executes the next statement; if the next statement contains a function call, then enters the function.
`next`	Executes the next statement.
`finish`	Enter the debug-mode again after finishing the execution of the current function. This is useful when an unnecessary `step` has been executed.
`cont`	
`quit`	Exits from the debug-mode and continues execution.
`up [n]`	Move up the call stack one level. Move up the call stack n levels if n is specified.
`down [n]`	Move down the call stack one level. Move down the call stack n levels if n is specified.
`frame [n]`	Print the current stack frame with no argument. n specifies the stack frame number to be selected. Here the stack frame number is a number at the top of lines displayed by executing `where`.

list [*startline*]
list *function*
> Displays ten lines in a source file from *startline*, the current line if the *startline* is not specified, or from the top line of current target *function*.

print *expr*
> Displays *expr*.

func *function*
> Set the target function to *function*.

stop at *sourceline* [if *cond*]
stop in *function*
> Set a break-point at the *sourceline*-th line of the source file, or at the top of the target function. Break-points are removed whenever the relevant function is redefined. When `if` statements are repeatedly encountered, **Asir** enters debug-mode only when the corresponding *cond* parts are evaluated to a non-zero value.

trace *expr* at *sourceline* [if *cond*]
trace *expr* in *function*
> These are similar to `stop`. `trace` simply displays the value of *expr* and without entering the debug-mode.

delete *n*   Remove the break point specified by a number *n*, which can be known by the `status` command.

status   Displays a list of the break-points.

where   Displays the calling sequence of functions from the top level through the current level.

alias *alias command*
> Create an alias *alias* for *command*

The debugger command `print` can take almost all expressions as its argument. The ordinary usage is to print the values of (programming) variables. However, the following usage is worth to remember.

- **overwriting the variable**

  One might sometimes wish to continue the execution with several values of variables modified. For such an purpose, take the following procedure.

  ```
  (debug) print A
  A = 2
  (debug) print A=1
  A=1 = 1
  (debug) print A
  A = 1
  ```

- **function call**

  A function call is also an expression, therefore, it can appear at the argument place of `print`.

  ```
  (debug) print length(List)
  length(List) = 14
  ```

In this example, the length of the list assigned to the variable `List` is examined by a function `length()`.

```
(debug) print ctrl("cputime",1)
ctrl("cputime",1) = 1
```

This example shows such a usage where measuring CPU time is activated from within the debug-mode, even if one might have forgotten to specify the activation of CPU time measurement.

It is also useful to save intermediate results to files from within the debug-mode by the built-in function `bsave()` when one is forced to quit the computation by any reason.

```
(debug) print bsave(A,"savefile")
bsave(A,"savefile") = 1
```

Note that continuation of the parent function will be impossible if an error will occur in the function call from within the debug-mode.

5.3 Execution example of debugger

Here, the usage of the Debugger is explained by showing an example for debugging a program which computes the integer factorial by a recursive definition.

```
% asir
[0] load("fac")$
[3] debug$
(debug) list factorial
1   def factorial(X) {
2       if ( !X )
3           return 1;
4       else
5           return X * factorial(X - 1);
6   }
7   end$
(debug) stop at 5                      <-- setting a break point
(0) stop at "./fac":5
(debug) quit                           <-- leaving the debug-mode
[4] factorial(6);                      <-- call for factorial(6)
stopped in factorial at line 5 in file "./fac"
5           return X * factorial(X - 1);
(debug) where                          <-- display the calling sequence
factorial(), line 5 in "./fac"             up to this break point
(debug) print X                        <-- Display the value of X
X = 6
(debug) step                           <-- step execution
                                           (enters function)
stopped in factorial at line 2 in file "./fac"
2       if ( !X )
(debug) where
factorial(), line 2 in "./fac"
factorial(), line 5 in "./fac"
```

```
(debug) print X
X = 5
(debug) delete 0                          <-- delete the break point 0
(debug) cont                              <-- continue execution
720                                       <-- result = 6!
[5] quit;
```

5.4 Sample file of initialization file for Debugger

As is previously mentioned, **Asir** reads in the file '`$HOME/.dbxinit`' at its invocation. This file is originally used to define various initializing commands for `dbx` debugger, but **Asir** recognizes only `alias` lines. For example, by the setting

```
% cat ~/.dbxinit
alias n next
alias c cont
alias p print
alias s step
alias d delete
alias r run
alias l list
alias q quit
```

one can use short aliases, e.g., `p`, `c` etc., for frequently used commands such as `print`, `cont` etc. One can create new aliases in the debug-mode during an execution.

```
lex_hensel(La,[a,b,c],0,[a,b,c],0);
stopped in gennf at line 226 in file "/home/usr3/noro/asir/gr"
226         N = length(V); Len = length(G); dp_ord(O); PS = newvect(Len);
(debug) p V
V = [a,b,c]
(debug) c
...
```

6 Built-in Function

6.1 Numbers

6.1.1 `idiv`, `irem`

`idiv(i1,i2)`
:: Integer quotient of *i1* divided by *i2*.

`irem(i1,i2)`
:: Integer remainder of *i1* divided by *i2*.

return   integer

i1 i2   integer

- Integer quotient and remainder of *i1* divided by *i2*.
- *i2* must not be 0.
- If the dividend is negative, the results are obtained by changing the sign of the results for absolute values of the dividend.
- One can use *i1* `%` *i2* for replacement of `irem()` which only differs in the point that the result is always normalized to non-negative values.
- Use `sdiv()`, `srem()` for polynomial quotient.

```
[0] idiv(100,7);
14
[0] idiv(-100,7);
-14
[1] irem(100,7);
2
[1] irem(-100,7);
-2
```

References
:   Section 6.3.8 [sdiv sdivm srem sremm sqr sqrm], page 50, Section 6.3.10 [%], page 52.

6.1.2 `fac`

`fac(i)`   :: The factorial of *i*.

return   integer

i   integer

- The factorial of *i*.
- Returns 0 if the argument *i* is negative.

```
[0] fac(50);
30414093201713378043612608166064768844377641568960512000000000000
```

6.1.3 `igcd,igcdcntl`

`igcd(`*i1*`,`*i2*`)`

:: The integer greatest common divisor of *i1* and *i2*.

`igcdcntl([`*i*`])`

:: Selects an algorithm for integer GCD.

return   integer

i1 i2 i   integer

- Function `igcd()` returns the integer greatest common divisor of the given two integers.
- An error will result if the argument is not an integer; the result is not valid even if one is returned.
- Use `gcd()`, `gcdz()` for polynomial GCD.
- Various method of integer GCD computation are implemented and they can be selected by `igcdcntl`.

    0   Euclid algorithm (default)

    1   binary GCD

    2   bmod GCD

    3   accelerated integer GCD

    2, 3 are due to `[Weber]`.

    In most cases 3 is the fastest, but there are exceptions.

    ```
    [0] A=lrandom(10^4)$
    [1] B=lrandom(10^4)$
    [2] C=lrandom(10^4)$
    [3] D=A*C$
    [4] E=A*B$
    [5] cputime(1)$
    [6] igcd(D,E)$
    0.6sec + gc : 1.93sec(2.531sec)
    [7] igcdcntl(1)$
    [8] igcd(D,E)$
    0.27sec(0.2635sec)
    [9] igcdcntl(2)$
    [10] igcd(D,E)$
    0.19sec(0.1928sec)
    [11] igcdcntl(3)$
    [12] igcd(D,E)$
    0.08sec(0.08023sec)
    ```

`References`

Section 6.3.20 [`gcd gcdz`], page 58.

6.1.4 `ilcm`

`ilcm(`*i1*`,`*i2*`)`
>> :: The integer least common multiple of *i1* and *i2*.

return   integer

i1 i2   integer

- This function computes the integer least common multiple of *i1*, *i2*.
- If one of argument is equal to 0, the return 0.

`References`
>> Section 6.1.3 [`igcd igcdcntl`], page 36, Section 6.1.10 [`mt_save mt_load`], page 39.

6.1.5 `isqrt`

`isqrt(`*n*`)`   :: The integer square root of *n*.

return   non-negative integer

n   non-negative integer

6.1.6 `inv`

`inv(`*i*`,`*m*`)`   :: the inverse (reciprocal) of *i* modulo *m*.

return   integer

i m   integer

- This function computes an integer such that $ia \equiv 1 \bmod (m)$.
- The integer *i* and *m* must be mutually prime. However, `inv()` does not check it.

```
[71] igcd(1234,4321);
1
[72] inv(1234,4321);
3239
[73] irem(3239*1234,4321);
1
```

`References`
>> Section 6.1.3 [`igcd igcdcntl`], page 36.

6.1.7 `prime, lprime`

`prime(`*index*`)`
`lprime(`*index*`)`
>> :: Returns a prime number.

return   integer

index   integer

- The two functions, `prime()` and `lprime()`, returns an element stored in the system table of prime numbers. Here, `index` is a non-negative integer and be used as an index

for the prime tables. The function `prime()` can return one of 1900 primes up to 16381 indexed so that the smaller one has smaller index. The function `lprime()` can return one of 999 primes which are 8 digit sized and indexed so that the larger one has the smaller index. The two function always returns 0 for other indices.

- For more general function for prime generation, there is a `PARI` function

  `pari(nextprime,number)`.

  ```
  [95] prime(0);
  2
  [96] prime(1228);
  9973
  [97] lprime(0);
  99999989
  [98] lprime(999);
  0
  ```

References
           Section 6.1.14 [`pari`], page 41.

6.1.8 `random`

`random([seed])`

seed

return   non-negative integer

- Generates a random number which is a non-negative integer less than 2^32.
- If a non zero argument is specified, then after setting it as a random seed, a random number is generated.
- As the default seed is fixed, the sequence of the random numbers is always the same if a seed is not set.
- The algorithm is Mersenne Twister (http://www.math.keio.ac.jp/matsumoto/mt.html) by M. Matsumoto and T. Nishimura. The implementation is done also by themselves.
- The period of the random number sequence is 2^19937-1.
- One can save the state of the random number generator with `mt_save`. By loading the state file with `mt_load`, one can trace a single random number sequence arcoss multiple sessions.

References
           Section 6.1.9 [`lrandom`], page 38, Section 6.1.10 [`mt_save mt_load`], page 39.

6.1.9 `lrandom`

`lrandom(bit)`
           :: Generates a long random number.

bit

return   integer

- Generates a non-negative integer of at most *bit* bits.
- The result is a concatenation of outputs of `random`.

References
        Section 6.1.8 [random], page 38, Section 6.1.10 [mt_save mt_load], page 39.

6.1.10 mt_save, mt_load

mt_save(*fname*)
        :: Saves the state of the random number generator.

mt_load(*fname*)
        :: Loads a saved state of the random number generator.

return   0 or 1

fname   string

- One can save the state of the random number generator with mt_save. By loading the state file with mt_load, one can trace a single random number sequence arcoss multiple **Asir** sessions.

```
[340] random();
3510405877
[341] mt_save("/tmp/mt_state");
1
[342] random();
4290933890
[343] quit;
% asir
This is Asir, Version 991108.
Copyright (C) FUJITSU LABORATORIES LIMITED.
3 March 1994. All rights reserved.
[340] mt_load("/tmp/mt_state");
1
[341] random();
4290933890
```

References
        Section 6.1.8 [random], page 38, Section 6.1.9 [lrandom], page 38.

6.1.11 nm, dn

nm(*rat*)   :: Numerator of *rat*.

dn(*rat*)   :: Denominator of *rat*.

return   integer or polynomial

rat   rational number or rational expression

- Numerator and denominator of a given rational expression.

- For a rational number, they return its numerator and denominator, respectively. For a rational expression whose numerator and denominator may contain rational numbers, they do not separate those rational coefficients to numerators and denominators.

- For a rational number, the denominator is always kept positive, and the sign is contained in the numerator.

- **Risa/Asir** does not cancel the common divisors unless otherwise explicitly specified by the user. Therefore, `nm()` and `dn()` return the numerator and the denominator as it is, respectively.

```
[2] [nm(-43/8),dn(-43/8)];
[-43,8]
[3] dn((x*z)/(x*y));
y*x
[3] dn(red((x*z)/(x*y)));
y
```

References
          Section 6.3.21 [red], page 59.

6.1.12 `conj, real, imag`

`real(`*comp*`)`
          :: Real part of *comp*.

`imag(`*comp*`)`
          :: Imaginary part of *comp*.

`conj(`*comp*`)`
          :: Complex conjugate of *comp*.

return comp
          complex number

- Basic operations for complex numbers.
- These functions works also for polynomials with complex coefficients.

```
[111] A=(2+@i)^3;
(2+11*@i)
[112] [real(A),imag(A),conj(A)];
[2,11,(2-11*@i)]
```

6.1.13 `eval, deval`

`eval(`*obj*`[,`*prec*`])`
`deval(`*obj*`)`
          :: Evaluate *obj* numerically.

return   number or expression

obj   general expression

prec   integer

- Evaluates the value of the functions contained in *obj* as far as possible.
- `deval` returns double float. Rational numbers remain unchanged in results from `eval`.
- In `eval` the computation is done by **MPFR** library. In `deval` the computation is done by the C math library.
- `deval` cannot handle complex numbers.

- When *prec* is specified, computation will be performed with a precision of about *prec*-digits. If *prec* is not specified, computation is performed with the precision set currently. (See Section 6.1.15 [setbprec setprec], page 42.)

- Currently available numerical functions are listed below.

  > `sin`, `cos`, `tan`,
  >
  > `asin`, `acos`, `atan`,
  >
  > `sinh`, `cosh`, `tanh`, `asinh`, `acosh`, `atanh`,
  >
  > `exp`, `log`, `pow(a,b)` `(a^b)`

- Symbols for special values are as follows. Note that `@i` cannot be handled by `deval`.

  `@i`   unit of imaginary number

  `@pi`   the number pi, the ratio of circumference to diameter

  `@e`   Napier's number (`exp(1)`)

  ```
  [118] eval(exp(@pi*@i));
  -1.0000000000000000000000000000000
  [119] eval(2^(1/2));
  1.4142135623730950488016887242097
  [120] eval(sin(@pi/3));
  0.86602540378443864676420506632
  [121] eval(sin(@pi/3)-3^(1/2)/2,50);
  -2.7879108444817914847 E-58
  [122] eval(1/2);
  1/2
  [123] deval(sin(1)^2+cos(1)^2);
  1
  ```

References
          Section 6.14.1 [ctrl], page 91, Section 6.1.15 [setbprec setprec], page 42.

6.1.14 pari

`pari(func,arg,prec)`
          :: Call **PARI** function *func*.

return   Depends on *func*.

func   Function name of **PARI**.

arg   Arguments of *func*.

prec   integer

- This command connects **Asir** to **PARI** system so that several functions of **PARI** can be conveniently used from **Risa/Asir**.

- **PARI** [Batut et al.] is developed at Bordeaux University, and distributed as a free software. Though it has a certain facility to computer algebra, its major target is the operation of numbers (**bignum**, **bigfloat**) related to the number theory. It facilitates various function evaluations as well as arithmetic operations at a remarkable speed. It can also be used from other external programs as a library. It provides a language

interface named 'gp' to its library, which enables a user to use **PARI** as a calculator which runs on UNIX.

- The last argument (optional) *int* specifies the precision in digits for bigfloat operation. If the precision is not explicitly specified, operation will be performed with the precision set by `setprec()`.

- Some of currently available functions of **PARI** system are as follows. Note these are only a part of functions in **PARI** system. A complete list of functions which can be called from asir is in the ox_pari manual of the asir-contrib. For details of individual functions, refer to the **PARI** manual. (Some of them can be seen in the following example.)

  ```
  abs, adj, arg, bigomega, binary, ceil, centerlift, cf, classno, classno2,
  conj, content, denom, det, det2, detr, dilog, disc, discf, divisors,
  eigen, eintg1, erfc, eta, floor, frac, galois, galoisconj, gamh, gamma,
  hclassno, hermite, hess, imag, image, image2, indexrank, indsort, initalg,
  isfund, isprime, ispsp, isqrt, issqfree, issquare, jacobi, jell, ker,
  keri, kerint, kerintg1, kerint2, kerr, length, lexsort, lift, lindep, lll,
  lllg1, lllgen, lllgram, lllgramg1, lllgramgen, lllgramint, lllgramkerim,
  lllgramkerimgen, lllint, lllkerim, lllkerimgen, lllrat, lngamma, logagm, mat,
  matrixqz2, matrixqz3, matsize, modreverse, mu, nextprime, norm, norml2, numdiv,
  numer, omega, order, ordred, phi, pnqn, polred, polred2, primroot, psi, quadgen,
  quadpoly, real, recip, redcomp, redreal, regula, reorder, reverse, rhoreal,
  roots, rootslong, round, sigma, signat, simplify, smalldiscf, smallfact,
  smallpolred, smallpolred2, smith, smith2, sort, sqr, sqred, sqrt, supplement,
  trace, trans, trunc, type, unit, vec, wf, wf2, zeta
  ```

- **Asir** currently uses only a very small subset of **PARI**.

  ```
  /* Eigen vectors of a numerical matrix */
  [0] pari(eigen,newmat(2,2,[[1,1],[1,2]]));
  [ -1.6180339887498948481977192199O O.618O339887498948482O ]
  [ 1 1 ]
  /* Roots of a polynomial */
  [1] pari(roots,t^2-2);
  [ -1.4142135623730950488 1.4142135623730950488 ]
  ```

```
References
        Section 6.1.15 [setbprec setprec], page 42.
```

6.1.15 setbprec, setprec

```
setbprec([n])
setprec([n])
```

　　　　　　:: **setbprec**, **setprec** set the precision for **bigfloat** operations to n bits, n digits respectively.

return　　　integer

n　　　　　integer

- When an argument n is given, these functions set the precision for **bigfloat** operations to n bits or n digits. The return value is always the previous precision regardless of the existence of an argument.

- **Bigfloat** operations are done by **MPFR** library.
- This is effective for computations in **bigfloat**. Refer to `ctrl()` for turning on the '**bigfloat** flag.'
- There is no upper limit for precision digits. It sets the precision to some digits around the specified precision. Therefore, it is safe to specify a larger value.

```
[1] setprec();
15
[2] setprec(100);
15
[3] setprec(100);
99
[4] setbprec();
332
```

6.1.16 `setround`

`setround([`*mode*`])`
          :: Sets the rounding mode *mode*.

return   integer

mode   integer

- When an argument *mode* is given, these functions set the rounding mode for **bigfloat** operations to *mode*. The return value is always the previous rounding mode regardless of the existence of an argument. The meanings of rounding modes are as follows

  0   Round to nearest

  1   Round toward 0

  2   Round toward +infinity

  3   Round toward -infinity

- This is effective for computations in **bigfloat**. Refer to `ctrl()` for turning on the '**bigfloat** flag.'

```
[1] setprec();
15
[2] setprec(100);
15
[3] setprec(100);
99
[4] setbprec();
332
```

6.1.17 `setmod`

`setmod([`*p*`])`
          :: Sets the ground field to GF(p).

return   integer

n   prime less than 2^27

- Sets the ground field to GF(p) and returns the value p.
- A member of a finite field does not have any information about the field and the arithmetic operations over GF(p) are applied with p set at the time.
- As for large finite fields, see Chapter 10 [Finite fields], page 176.

```
[0] A=dp_mod(dp_ptod(2*x,[x]),3,[]);
(2)*<<1>>
[1] A+A;
addmi : invalid modulus
return to toplevel
[1] setmod(3);
3
[2] A+A;
(1)*<<1>>
```

References
        Section 8.10.18 [dp_mod dp_rat], page 144, Section 3.2 [Types of numbers], page 14.

6.1.18 ntoint32, int32ton

`ntoint32(`*n*`)`
`int32ton(`*int32*`)`
        :: Type-conversion between a non-negative integer and an unsigned 32bit integer.

return   unsigned 32bit integer or non-negative integer

n   non-negative integer less than 2^32

int32   unsigned 32bit integer

- These functions do conversions between non-negative integers (the type id 1) and unsigned 32bit integers (the type id 10).
- An unsigned 32bit integer is a fundamental construct of **OpenXM** and one often has to send an integer to a server as an unsigned 32bit integer. These functions are used in such a case.

References
        Chapter 7 [Distributed computation], page 102, Section 3.2 [Types of numbers], page 14.

6.1.19 inttorat

`inttorat(`*a*`,`*m*`,`*b*`)`
        :: Perform the rational reconstruction.

return   list or 0

a

m

b   integer

- For an integer *a*, find a positive integer *x* and an integer *y* satisfying *xa=y* mod *m*, *x*, |*y*| < *b* and *GCD(x,y)=1*.

- If such *x*, *y* exist then a list *[y,x]* is returned. Otherwise 0 is returned.

- If *b* is set to *floor(sqrt(M/2))*, then *x* and *y* are unique if they exist. *floor(sqrt(M/2))* can be computed by `floor` and `isqrt`.

```
[2121] M=lprime(0)*lprime(1);
9996359931312779
[2122] B=isqrt(floor(M/2));
70697807
[2123] A=234234829304;
234234829304
[2124] inttorat(A,M,B);
[-20335178,86975031]
```

`References`
          Section 6.13.3 [`ceil floor rint dceil dfloor drint`], page 91, Section 6.1.5
          [`isqrt`], page 37.

6.2 Bit operations

6.2.1 `iand, ior, ixor`

`iand(`*i1*`,`*i2*`)`
          :: bitwise and

`ior(`*i1*`,`*i2*`)`
          :: bitwise or

`ixor(`*i1*`,`*i2*`)`
          :: bitwise xor

return   integer

i1 i2   integer

- The absolute value of the argument is regarded as a bit string.

- The sign of the argument is ignored and a non-negative integer is returned.

```
[0] ctrl("hex",1);
0x1
[1] iand(0xeeeeeeeeeeeeeeee,0x2984723234812312312);
0x4622224802202202
[2] ior(0xa0a0a0a0a0a0a0a0,0xb0c0b0b0b0b0b0b0b);
0xabacababababababab
[3] ixor(0xffffffffffff,0x234234234234);
0x2cbdcbdcbdcb
```

`References`
          Section 6.2.2 [`ishift`], page 46.

6.2.2 `ishift`

`ishift(i,count)`
            :: bit shift

return   integer

i count   integer

- The absolute value of *i* is regarded as a bit string.

- The sign of *i* is ignored and a non-negative integer is returned.

- If *count* is positive, then *i* is shifted to the right. If *count* is negative, then *i* is shifted to the left.

  ```
  [0] ctrl("hex",1);
  0x1
  [1] ishift(0x1000000,12);
  0x1000
  [2] ishift(0x1000,-12);
  0x1000000
  [3] ixor(0x1248,ishift(1,-16)-1);
  ```

`References`
            Section 6.2.1 [`iand ior ixor`], page 45.

6.3 operations with polynomials and rational expressions

6.3.1 `var`

`var(rat)`   :: Main variable (indeterminate) of *rat*.

return   indeterminate

rat   rational expression

- See Section 3.1 [Types in Asir], page 11, for main variable.

- Indeterminates (variables) are ordered by default as follows.

  x, y, z, u, v, w, p, q, r, s, t, a, b, c, d, e, f, g, h, i, j, k, l, m, n, o. The other variables will be ordered after the above noted variables so that the first comer will be ordered prior to the followers.

  ```
  [0] var(x^2+y^2+a^2);
  x
  [1] var(a*b*c*d*e);
  a
  [2] var(3/abc+2*xy/efg);
  abc
  ```

`References`
            Section 6.3.7 [`ord`], page 49, Section 6.3.2 [`vars`], page 47.

6.3.2 `vars`

`vars(obj)`
>> :: A list of variables (indeterminates) in an expression *obj*.

return   list

obj   arbitrary

- Returns a list of variables (indeterminates) contained in a given expression.
- Lists variables according to the variable ordering.

```
[0] vars(x^2+y^2+a^2);
[x,y,a]
[1] vars(3/abc+2*xy/efg);
[abc,xy,efg]
[2] vars([x,y,z]);
[x,y,z]
```

References
>> Section 6.3.1 [var], page 46, Section 6.3.3 [uc], page 47, Section 6.3.7 [ord], page 49.

6.3.3 `uc`

`uc()`   :: Create a new indeterminate for an undetermined coefficient.

return   indeterminate with its `vtype` 1.

- At every evaluation of command `uc()`, a new indeterminate in the sequence of indeterminates `_0`, `_1`, `_2`, . . . is created successively.
- Indeterminates created by `uc()` cannot be input on the keyboard. By this property, you are free, no matter how many indeterminates you will create dynamically by a program, from collision of created names with indeterminates input from the keyboard or from program files.
- Functions, `rtostr()` and `strtov()`, are used to create ordinary indeterminates (indeterminates having 0 for their `vtype`).
- Kernel sub-type of indeterminates created by `uc()` is 1. (`vtype(uc())=1`)

```
[0] A=uc();
_0
[1] B=uc();
_1
[2] (uc()+uc())^2;
_2^2+2*_3*_2+_3^2
[3] (A+B)^2;
_0^2+2*_1*_0+_1^2
```

References
>> Section 6.8.3 [vtype], page 78, Section 6.10.1 [rtostr], page 80, Section 6.10.2 [strtov], page 80.

6.3.4 `coef`

`coef(poly,deg[,var])`
> :: The coefficient of a polynomial *poly* at degree *deg* with respect to the variable *var* (main variable if unspecified).

return   polynomial

poly   polynomial

var   indeterminate

deg   non-negative integer

- The coefficient of a polynomial *poly* at degree *deg* with respect to the variable *var*.
- The default value for *var* is the main variable, i.e., `var(poly)`.
- For multi-variate polynomials, access to coefficients depends on the specified indeterminates. For example, taking coef for the main variable is much faster than for other variables.

```
[0] A = (x+y+z)^3;
x^3+(3*y+3*z)*x^2+(3*y^2+6*z*y+3*z^2)*x+y^3+3*z*y^2+3*z^2*y+z^3
[1] coef(A,1,y);
3*x^2+6*z*x+3*z^2
[2] coef(A,0);
y^3+3*z*y^2+3*z^2*y+z^3
```

`References`
> Section 6.3.1 [`var`], page 46, Section 6.3.5 [`deg mindeg`], page 48.

6.3.5 `deg`, `mindeg`

`deg(poly,var)`
> :: The degree of a polynomial *poly* with respect to variable.

`mindeg(poly,var)`
> :: The least exponent of the terms with non-zero coefficients in a polynomial *poly* with respect to the variable *var*. In this manual, this quantity is sometimes referred to the minimum degree of a polynomial for short.

return   non-negative integer

poly   polynomial

var   indeterminate

- The least exponent of the terms with non-zero coefficients in a polynomial *poly* with respect to the variable *var*. In this manual, this quantity is sometimes referred to the minimum degree of a polynomial for short.
- Variable *var* must be specified.

```
[0] deg((x+y+z)^10,x);
10
[1] deg((x+y+z)^10,w);
0
[75] mindeg(x^2+3*x*y,x);
1
```

6.3.6 `nmono`

`nmono(rat)`

:: Number of monomials in rational expression *rat*.

return non-negative integer

rat   rational expression

- Number of monomials with non-zero number coefficients in the full expanded form of the given polynomial.

- For a rational expression, the sum of the numbers of monomials of the numerator and denominator.

- A function form is regarded as a single indeterminate no matter how complex arguments it has.

```
[0] nmono((x+y)^10);
11
[1] nmono((x+y)^10/(x+z)^10);
22
[2] nmono(sin((x+y)^10));
1
```

`References`

Section 6.8.3 [vtype], page 78.

6.3.7 `ord`

`ord([varlist])`

:: It sets the ordering of indeterminates (variables).

return list of indeterminates

varlist list of indeterminates

- When an argument is given, this function rearranges the ordering of variables (indeterminates) so that the indeterminates in the argument *varlist* precede and the other indeterminates follow in the system's variable ordering. Regardless of the existence of an argument, it always returns the final variable ordering.

- Note that no change will be made to the variable ordering of internal forms of objects which already exists in the system, no matter what reordering you specify. Therefore, the reordering should be limited to the time just after starting **Asir**, or to the time when one has decided himself to start a totally new computation which has no relation with the previous results. Note that unexpected results may be obtained from operations between objects which are created under different variable ordering.

```
[0] ord();
[x,y,z,u,v,w,p,q,r,s,t,a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,_x,_y,_z,_u,_v,
_w,_p,_q,_r,_s,_t,_a,_b,_c,_d,_e,_f,_g,_h,_i,_j,_k,_l,_m,_n,_o,
exp(_x),(_x)^(_y),log(_x),(_x)^(_y-1),cos(_x),sin(_x),tan(_x),
(-_x^2+1)^(-1/2),cosh(_x),sinh(_x),tanh(_x),
(_x^2+1)^(-1/2),(_x^2-1)^(-1/2)]
[1] ord([dx,dy,dz,a,b,c]);
```

```
[dx,dy,dz,a,b,c,x,y,z,u,v,w,p,q,r,s,t,d,e,f,g,h,i,j,k,l,m,n,o,_x,_y,
_z,_u,_v,_w,_p,_q,_r,_s,_t,_a,_b,_c,_d,_e,_f,_g,_h,_i,_j,_k,_l,_m,_n,
_o,exp(_x),(_x)^(_y),log(_x),(_x)^(_y-1),cos(_x),sin(_x),tan(_x),
(-_x^2+1)^(-1/2),cosh(_x),sinh(_x),tanh(_x),
(_x^2+1)^(-1/2),(_x^2-1)^(-1/2)]
```

6.3.8 `sdiv`, `sdivm`, `srem`, `sremm`, `sqr`, `sqrm`

`sdiv(poly1,poly2[,v])`
`sdivm(poly1,poly2,mod[,v])`
>   :: Quotient of *poly1* divided by *poly2* provided that the division can be performed within polynomial arithmetic over the rationals.

`srem(poly1,poly2[,v])`
`sremm(poly1,poly2,mod[,v])`
>   :: Remainder of *poly1* divided by *poly2* provided that the division can be performed within polynomial arithmetic over the rationals.

`sqr(poly1,poly2[,v])`
`sqrm(poly1,poly2,mod[,v])`
>   :: Quotient and remainder of *poly1* divided by *poly2* provided that the division can be performed within polynomial arithmetic over the rationals.

return   `sdiv()`, `sdivm()`, `srem()`, `sremm()` : polynomial `sqr()`, `sqrm()` : a list `[quotient,remainder]`

poly1 poly2
>   polynomial

v   indeterminate

mod   prime

- Regarding *poly1* as an uni-variate polynomial in the main variable of *poly2*, i.e. `var(poly2)` (*v* if specified), `sdiv()` and `srem()` compute the polynomial quotient and remainder of *poly1* divided by *poly2*.
- `sdivm()`, `sremm()`, `sqrm()` execute the same operation over GF(*mod*).
- Division operation of polynomials is performed by the following steps: (1) obtain the quotient of leading coefficients; let it be Q; (2) remove the leading term of *poly1* by subtracting, from *poly1*, the product of Q with some powers of main variable and *poly2*; obtain a new *poly1*; (3) repeat the above step until the degree of *poly1* become smaller than that of *poly2*. For fulfillment, by operating in polynomials, of this procedure, the divisions at step (1) in every repetition must be an exact division of polynomials. This is the true meaning of what we say "division can be performed within polynomial arithmetic over the rationals."
- There are typical cases where the division is possible: leading coefficient of *poly2* is a rational number; *poly2* is a factor of *poly1*.
- Use `sqr()` to get both the quotient and remainder at once.
- Use `idiv()`, `irem()` for integer quotient.
- For remainder operation on all integer coefficients, use `%`.

```
[0] sdiv((x+y+z)^3,x^2+y+a);
x+3*y+3*z
[1] srem((x+y+z)^2,x^2+y+a);
(2*y+2*z)*x+y^2+(2*z-1)*y+z^2-a
[2] X=(x+y+z)*(x-y-z)^2;
x^3+(-y-z)*x^2+(-y^2-2*z*y-z^2)*x+y^3+3*z*y^2+3*z^2*y+z^3
[3] Y=(x+y+z)^2*(x-y-z);
x^3+(y+z)*x^2+(-y^2-2*z*y-z^2)*x-y^3-3*z*y^2-3*z^2*y-z^3
[4] G=gcd(X,Y);
x^2-y^2-2*z*y-z^2
[5] sqr(X,G);
[x-y-z,0]
[6] sqr(Y,G);
[x+y+z,0]
[7] sdiv(y*x^3+x+1,y*x+1);
divsp: cannot happen
return to toplevel
```

References
        Section 6.1.1 [idiv irem], page 35, Section 6.3.10 [%], page 52.

6.3.9 tdiv

tdiv(*poly1*,*poly2*)
        :: Tests whether *poly2* divides *poly1*.

return   Quotient if *poly2* divides *poly1*, 0 otherwise.

poly1 poly2
        polynomial

- Tests whether *poly2* divides *poly1* in polynomial ring.
- One application of this function: Consider the case where a polynomial is certainly an irreducible factor of the other polynomial, but the multiplicity of the factor is unknown. Application of tdiv() to the polynomials repeatedly yields the multiplicity.

```
[11] Y=(x+y+z)^5*(x-y-z)^3;
x^8+(2*y+2*z)*x^7+(-2*y^2-4*z*y-2*z^2)*x^6
+(-6*y^3-18*z*y^2-18*z^2*y-6*z^3)*x^5
+(6*y^5+30*z*y^4+60*z^2*y^3+60*z^3*y^2+30*z^4*y+6*z^5)*x^3
+(2*y^6+12*z*y^5+30*z^2*y^4+40*z^3*y^3+30*z^4*y^2+12*z^5*y+2*z^6)*x^2
+(-2*y^7-14*z*y^6-42*z^2*y^5-70*z^3*y^4-70*z^4*y^3-42*z^5*y^2
-14*z^6*y-2*z^7)*x-y^8-8*z*y^7-28*z^2*y^6-56*z^3*y^5-70*z^4*y^4
-56*z^5*y^3-28*z^6*y^2-8*z^7*y-z^8
[12] for(I=0,F=x+y+z,T=Y; T=tdiv(T,F); I++);
[13] I;
5
```

References
        Section 6.3.8 [sdiv sdivm srem sremm sqr sqrm], page 50.

6.3.10 %

`poly % m`   :: integer remainder to all integer coefficients of the polynomial.

return   integer or polynomial

poly   integer or polynomial with integer coefficients

m   integer

- Returns a polynomial whose coefficients are remainders of the coefficients of the input polynomial divided by *m*.
- The resulting coefficients are all normalized to non-negative integers.
- An integer is allowed for *poly*. This can be used for an alternative for `irem()` except that the result is normalized to a non-negative integer.
- Coefficients of *poly* and *m* must all be integers, though the type checking is not done.

```
[0] (x+2)^5 % 3;
x^5+x^4+x^3+2*x^2+2*x+2
[1] (x-2)^5 % 3;
x^5+2*x^4+x^3+x^2+2*x+1
[2] (-5) % 4;
3
[3] irem(-5,4);
-1
```

References
          Section 6.1.1 [`idiv irem`], page 35.

6.3.11 `subst, psubst`

```
subst(rat[,varn,ratn]*)
psubst(rat[,var,rat]*)
```
          :: Substitute *ratn* for *varn* in expression *rat*. (*n*=1,2,.... Substitution will be done successively from left to right if arguments are repeated.)

return   rational expression

rat ratn   rational expression

varn   indeterminate

- Substitutes rational expressions for specified kernels in a rational expression.
- `subst`(*r,v1,r1,v2,r2,*...) has the same effect as `subst`(`subst`(*r,v1,r1*)*,v2,r2,*...).
- Note that repeated substitution is done from left to right successively. You may get different result by changing the specification order.
- Ordinary `subst()` performs substitution at all levels of a scalar algebraic expression creeping into arguments of function forms recursively. Function `psubst()` regards such a function form as an independent indeterminate, and does not attempt to apply substitution to its arguments. (The name comes after Partial SUBSTitution.)
- Since **Asir** does not reduce common divisors of a rational expression automatically, substitution of a rational expression to an expression may cause unexpected increase of computation time. Thus, it is often necessary to write a special function to meet

the individual problem so that the denominator and the numerator do not become too large.

- The same applies to substitution by rational numbers.

```
[0] subst(x^3-3*y*x^2+3*y^2*x-y^3,y,2);
x^3-6*x^2+12*x-8
[1] subst(@@,x,-1);
-27
[2] subst(x^3-3*y*x^2+3*y^2*x-y^3,y,2,x,-1);
-27
[3] subst(x*y^3,x,y,y,x);
x^4
[4] subst(x*y^3,y,x,x,y);
y^4
[5] subst(x*y^3,x,t,y,x,t,y);
y*x^3
[6] subst(x*sin(x),x,t);
sint(t)*t
[7] psubst(x*sin(x),x,t);
sin(x)*t
```

6.3.12 `diff`

`diff(`*rat*`[,`*varn*`]*)`
`diff(`*rat*`,`*varlist*`)`

> :: Differentiate *rat* successively by *var*'s for the first form, or by variables in *varlist* for the second form.

return   expression

rat   rational expression which contains elementary functions.

varn   indeterminate

varlist   list of indeterminates

- Differentiate *rat* successively by *var*'s for the first form, or by variables in *varlist* for the second form.

- differentiation is performed by the specified indeterminates (variables) from left to right. `diff(`*rat*`,x,y)` is the same as `diff(diff(`*rat*`,x),y)`.

```
[0] diff((x+2*y)^2,x);
2*x+4*y
[1] diff((x+2*y)^2,x,y);
4
[2] diff(x/sin(log(x)+1),x);
(sin(log(x)+1)-cos(log(x)+1))/(sin(log(x)+1)^2)
[3] diff(sin(x),[x,x,x,x]);
sin(x)
```

6.3.13 `ediff`

`ediff(poly[,varn]*)`
`ediff(poly,varlist)`
>   :: Differentiate *poly* successively by Euler operators of *var*'s for the first form, or by Euler operators of variables in *varlist* for the second form.

return   polynomial

poly   polynomial

varn   indeterminate

varlist   list of indeterminates

- differentiation is performed by the specified indeterminates (variables) from left to right. `ediff(`*poly*`,x,y)` is the same as `ediff(ediff(`*poly*`,x),y)`.

   ```
   [0] ediff((x+2*y)^2,x);
   2*x^2+4*y*x
   [1] ediff((x+2*y)^2,x,y);
   4*y*x
   ```

6.3.14 `res`

`res(var,poly1,poly2[,mod])`
>   :: Resultant of *poly1* and *poly2* with respect to *var*.

return   polynomial

var   indeterminate

poly1 poly2
>   polynomial

mod   prime

- Resultant of two polynomials *poly1* and *poly2* with respect to *var*.
- Sub-resultant algorithm is used to compute the resultant.
- The computation is done over GF(*mod*) if *mod* is specified.

   ```
   [0] res(t,(t^3+1)*x+1,(t^3+1)*y+t);
   -x^3-x^2-y^3
   ```

6.3.15 `fctr, sqfr`

`fctr(poly)`
>   :: Factorize polynomial *poly* over the rationals.

`sqfr(poly)`
>   :: Gets a square-free factorization of polynomial *poly*.

return   list

poly   polynomial with rational coefficients

- Factorizes polynomial *poly* over the rationals. `fctr()` for irreducible factorization; `sqfr()` for square-free factorization.

- The result is represented by a list, whose elements are a pair represented as [[**num**,1],[**factor**,**multiplicity**],...].
- Products of all **factor^multiplicity** and **num** is equal to *poly*.
- The number **num** is determined so that (*poly*/**num**) is an integral polynomial and its content (GCD of all coefficients) is 1. (See Section 6.3.18 [ptozp], page 57.)

```
[0] fctr(x^10-1);
[[1,1],[x-1,1],[x+1,1],[x^4+x^3+x^2+x+1,1],[x^4-x^3+x^2-x+1,1]]
[1] fctr(x^3+y^3+(z/3)^3-x*y*z);
[[1/27,1],[9*x^2+(-9*y-3*z)*x+9*y^2-3*z*y+z^2,1],[3*x+3*y+z,1]]
[2] A=(a+b+c+d)^2;
a^2+(2*b+2*c+2*d)*a+b^2+(2*c+2*d)*b+c^2+2*d*c+d^2
[3] fctr(A);
[[1,1],[a+b+c+d,2]]
[4] A=(x+1)*(x^2-y^2)^2;
x^5+x^4-2*y^2*x^3-2*y^2*x^2+y^4*x+y^4
[5] sqfr(A);
[[1,1],[x+1,1],[-x^2+y^2,2]]
[6] fctr(A);
[[1,1],[x+1,1],[-x-y,2],[x-y,2]]
```

References
          Section 6.3.16 [`ufctrhint`], page 55.

6.3.16 `ufctrhint`

`ufctrhint(poly,hint)`
          :: Factorizes uni-variate polynomial *poly* over the rational number field when the degrees of its factors are known to be some integer multiples of *hint*.

return   list

poly   uni-variate polynomial with rational coefficients

hint   non-negative integer

- By any reason, if the degree of all the irreducible factors of *poly* is known to be some multiples of *hint*, factors can be computed more efficiently by the knowledge than `fctr()`.
- When *hint* is 1, `ufctrhint()` is the same as `fctr()` for uni-variate polynomials. An typical application where `ufctrhint()` is effective: Consider the case where *poly* is a norm (Chapter 9 [Algebraic numbers], page 162) of a certain polynomial over an extension field with its extension degree *d*, and it is square free; Then, every irreducible factor has a degree that is a multiple of *d*.

```
[10] A=t^9-15*t^6-87*t^3-125;
t^9-15*t^6-87*t^3-125
0msec
[11] N=res(t,subst(A,t,x-2*t),A);
-x^81+1215*x^78-567405*x^75+139519665*x^72-19360343142*x^69
+1720634125410*x^66-88249977024390*x^63-4856095669551930*x^60
```

```
        +1999385245240571421*x^57-15579689952590251515*x^54
        +15956967531741971462865*x^51
        ...
        +14039558872035397353552612361266144550659875*x^6
        +1012232428734315543004276892350079984375*x^3
        +13926274344440731013345902118273314453125
        980msec + gc : 250msec
        [12] sqfr(N);
        [[-1,1],[x^81-1215*x^78+567405*x^75-139519665*x^72+19360343142*x^69
        -1720634125410*x^66+88249977024390*x^63+4856095669551930*x^60
        -1999385245240571421*x^57+15579689952590251515*x^54
        ...
        -1012232428734315543004276892350079984375*x^3
        -13926274344440731013345902118273314453125,1]]
        20msec
        [13] fctr(N);
        [[-1,1],[x^9-405*x^6-63423*x^3-2460375,1],
        [x^18-486*x^15+98739*x^12-9316620*x^9+945468531*x^6-12368049246*x^3
        +296607516309,1],[x^18-8667*x^12+19842651*x^6+19683,1],
        [x^18-324*x^15+44469*x^12-1180980*x^9+427455711*x^6+2793253896*x^3
        +31524548679,1],
        [x^18+10773*x^12+2784051*x^6+307546875,1]]
        167.050sec + gc : 1.890sec
        [14] ufctrhint(N,9);
        [[-1,1],[x^9-405*x^6-63423*x^3-2460375,1],
        [x^18-486*x^15+98739*x^12-9316620*x^9+945468531*x^6-12368049246*x^3
        +296607516309,1],[x^18-8667*x^12+19842651*x^6+19683,1],
        [x^18-324*x^15+44469*x^12-1180980*x^9+427455711*x^6+2793253896*x^3
        +31524548679,1],
        [x^18+10773*x^12+2784051*x^6+307546875,1]]
        119.340sec + gc : 1.300sec
```

References
        Section 6.3.15 [`fctr sqfr`], page 54.

6.3.17 `modfctr`

`modfctr(`*poly*`,`*mod*`)`
        :: Factorizer over small finite fields

return   list

poly   Polynomial with integer coefficients

mod   non-negative integer

- This function factorizes a polynomial *poly* over the finite prime field of characteristic *mod*, where *mod* must be smaller than 2^29.
- The result is represented by a list, whose elements are a pair represented as
  [[**num**,1],[**factor**,**multiplicity**],...].

- Products of all **factor^multiplicity** and **num** is equal to *poly*.
- To factorize polynomials over large finite fields, use `fctr_ff` (see Chapter 10 [Finite fields], page 176,Section 10.5.16 [fctr_ff], page 186).

```
[0] modfctr(x^10+x^2+1,2147483647);
[[1,1],[x+1513477736,1],[x+2055628767,1],[x+91854880,1],
[x+634005911,1],[x+1513477735,1],[x+634005912,1],
[x^4+1759639395*x^2+2045307031,1]]
[1] modfctr(2*x^6+(y^2+z*y)*x^4+2*z*y^3*x^2+(2*z^2*y^2+z^3*y)*x+z^4,3);
[[2,1],[2*x^3+z*y*x+z^2,1],[2*x^3+y^2*x+2*z^2,1]]
```

References

Section 6.3.15 [`fctr sqfr`], page 54.

6.3.18 `ptozp`

`ptozp(poly)`

:: Converts a polynomial *poly* with rational coefficients into an integral polynomial such that GCD of all its coefficients is 1.

return   polynomial

poly   polynomial

- Converts the given polynomial by multiplying some rational number into an integral polynomial such that GCD of all its coefficients is 1.
- In general, operations on polynomials can be performed faster for integer coefficients than for rational number coefficients. Therefore, this function is conveniently used to improve efficiency.
- Function `red` does not convert rational coefficients of the numerator. You cannot obtain an integral polynomial by direct use of the function `nm()`. The function `nm()` returns the numerator of its argument, and a polynomial with rational coefficients is the numerator of itself and will be returned as it is.
- When the option factor is set, the return value is a list [g,c]. Here, c is a rational number, g is an integral polynomial and *poly* = c*g holds.

```
[0] ptozp(2*x+5/3);
6*x+5
[1] nm(2*x+5/3);
2*x+5/3
```

References

Section 6.1.11 [`nm dn`], page 39.

6.3.19 `prim, cont`

`prim(poly[,v])`

:: Primitive part of *poly*.

`cont(poly[,v])`

:: Content of *poly*.

return poly

polynomial over the rationals

v   indeterminate

- The primitive part and the content of a polynomial *poly* with respect to its main variable (*v* if specified).

```
[0] E=(y-z)*(x+y)*(x-z)*(2*x-y);
(2*y-2*z)*x^3+(y^2-3*z*y+2*z^2)*x^2+(-y^3+z^2*y)*x+z*y^3-z^2*y^2
[1] prim(E);
2*x^3+(y-2*z)*x^2+(-y^2-z*y)*x+z*y^2
[2] cont(E);
y-z
[3] prim(E,z);
(y-z)*x-z*y+z^2
```

References
         Section 6.3.1 [var], page 46, Section 6.3.7 [ord], page 49.

6.3.20 gcd, gcdz

```
gcd(poly1,poly2[,mod])
gcdz(poly1,poly2)
```
         :: The polynomial greatest common divisor of *poly1* and *poly2*.

return   polynomial

poly1 poly2
         polynomial

mod   prime

- Functions gcd() and gcdz() return the greatest common divisor (GCD) of the given two polynomials.

- Function gcd() returns an integral polynomial GCD over the rational number field. The coefficients are normalized such that their GCD is 1. It returns 1 in case that the given polynomials are mutually prime.

- Function gcdz() works for arguments of integral polynomials, and returns a polynomial GCD over the integer ring, that is, it returns gcd() multiplied by the contents of all coefficients of the two input polynomials.

- gcd() computes the GCD over GF(*mod*) if *mod* is specified.

- Polynomial GCD is computed by an improved algorithm based on Extended Zassenhaus algorithm.

- GCD over a finite field is computed by PRS algorithm and it may not be efficient for large inputs and co-prime inputs.

```
[0] gcd(12*(x^2+2*x+1)^2,18*(x^2+(y+1)*x+y)^3);
x^3+3*x^2+3*x+1
[1] gcdz(12*(x^2+2*x+1)^2,18*(x^2+(y+1)*x+y)^3);
6*x^3+18*x^2+18*x+6
[2] gcd((x+y)*(x-y)^2,(x+y)^2*(x-y));
x^2-y^2
[3] gcd((x+y)*(x-y)^2,(x+y)^2*(x-y),2);
```

```
        x^3+y*x^2+y^2*x+y^3
```
References
        Section 6.1.3 [`igcd igcdcntl`], page 36.

6.3.21 `red`

`red(rat)`   :: Reduced form of *rat* by canceling common divisors.

return   rational expression

rat   rational expression

- **Asir** automatically performs cancellation of common divisors of rational numb ers. But, without an explicit command, it does not cancel common polynomial divisors of rational expressions. (Reduction of rational expressions to a common denominator will be always done.) Use command `red()` to perform this cancellation.
- Cancel the common divisors of the numerator and the denominator of a rational expression *rat* by computing their GCD.
- The denominator polynomial of the result is an integral polynomial which has no common divisors in its coefficients, while the numerator may have rational coefficients.
- Since GCD computation is a very hard operation, it is desirable to detect and remove by any means common divisors as far as possible. Furthermore, a call to this function after swelling of the denominator and the numerator shall usually take a very long time. Therefore, often, to some extent, reduction of common divisors is inevitable for operations of rational expressions.

```
    [0] (x^3-1)/(x-1);
    (x^3-1)/(x-1)
    [1] red((x^3-1)/(x-1));
    x^2+x+1
    [2] red((x^3+y^3+z^3-3*x*y*z)/(x+y+z));
    x^2+(-y-z)*x+y^2-z*y+z^2
    [3] red((3*x*y)/(12*x^2+21*y^3*x));
    (y)/(4*x+7*y^3)
    [4] red((3/4*x^2+5/6*x)/(2*y*x+4/3*x));
    (9/8*x+5/4)/(3*y+2)
```

References
        Section 6.1.11 [nm dn], page 39, Section 6.3.20 [`gcd gcdz`], page 58, Section 6.3.18 [`ptozp`], page 57.

6.4 Univariate polynomials

6.4.1 `umul, umul_ff, usquare, usquare_ff, utmul, utmul_ff`

`umul(p1,p2)`
`umul_ff(p1,p2)`
        :: Fast multiplication of univariate polynomials

`usquare(p1)`
`usquare_ff(p1)`
        :: Fast squaring of a univariate polynomial

```
utmul(p1,p2,d)
utmul_ff(p1,p2,d)
```
               :: Fast multiplication of univariate polynomials with truncation

return   univariate polynomial

p1 p2   univariate polynomial

d   non-negative integer

- These functions compute products of univariate polynomials by selecting an appropriate algorithm depending on the degrees of inputs.

- `umul()`, `usquare()`, `utmul()` compute products over the integers. Coefficients in GF(p) are regarded as non-negative integers less than p.

- `umul_ff()`, `usquare_ff()`, `utmul_ff()` compute products over a finite field. However, if some of the coefficients of the inputs are integral, the result may be an integral polynomial. So if one wants to assure that the result is a polynomial over the finite field, apply `simp_ff()` to the inputs.

- `umul_ff()`, `usquare_ff()`, `utmul_ff()` cannot take polynomials over GF(2^n) as their inputs.

- `umul()`, `umul_ff()` produce *p1*p2*. `usquare()`, `usquare_ff()` produce *p1^2*. `utmul()`, `utmul_ff()` produce *p1*p2 mod v^(d+1)*, where *v* is the variable of *p1, p2*.

- If the degrees of the inputs are less than or equal to the value returned by `set_upkara()` (`set_uptkara()` for utmul, utmul_ff), usual pencil and paper method is used. If the degrees of the inputs are less than or equal to the value returned by `set_upfft()`, Karatsuba algorithm is used. If the degrees of the inputs exceed it, a combination of FFT and Chinese remainder theorem is used. First of all sufficiently many primes *mi* within 1 machine word are prepared. Then *p1*p2 mod mi* is computed by FFT for each *mi*. Finally they are combined by Chinese remainder theorem. The functions over finite fields use an improvement by V. Shoup [`Shoup`].

```
[176] load("fff")$
[177] cputime(1)$
0sec(1.407e-05sec)
[178] setmod_ff(2^160-47);
1461501637330902918203684832716283019655932542929
0sec(0.00028sec)
[179] A=randpoly_ff(100,x)$
0sec(0.001422sec)
[180] B=randpoly_ff(100,x)$
0sec(0.00107sec)
[181] for(I=0;I<100;I++)A*B;
7.77sec + gc : 8.38sec(16.15sec)
[182] for(I=0;I<100;I++)umul(A,B);
2.24sec + gc : 1.52sec(3.767sec)
[183] for(I=0;I<100;I++)umul_ff(A,B);
1.42sec + gc : 0.24sec(1.653sec)
[184] for(I=0;I<100;I++)usquare_ff(A);
1.08sec + gc : 0.21sec(1.297sec)
```

```
[185] for(I=0;I<100;I++)utmul_ff(A,B,100);
1.2sec + gc : 0.17sec(1.366sec)
[186] deg(utmul_ff(A,B,100),x);
100
```

References

Section 6.4.3 [set_upkara set_uptkara set_upfft], page 61, Section 6.4.2 [kmul ksquare ktmul], page 61.

6.4.2 kmul, ksquare, ktmul

kmul(*p1,p2*)

:: Fast multiplication of univariate polynomials

ksquare(*p1*)

:: Fast squaring of a univariate polynomial

ktmul(*p1,p2,d*)

:: Fast multiplication of univariate polynomials with truncation

return   univariate polynomial

p1 p2   univariate polynomial

d   non-negative integer

These functions compute products of univariate polynomials by Karatsuba algorithm.

- These functions do not apply FFT for large degree inputs.
- These functions can compute products over GF(2^n).

```
[0] load("code/fff");
1
[34] setmod_ff(defpoly_mod2(160));
x^160+x^5+x^3+x^2+1
[35] A=randpoly_ff(100,x)$
[36] B=randpoly_ff(100,x)$
[37] umul(A,B)$
umul : invalid argument
return to toplevel
[37] kmul(A,B)$
```

6.4.3 set_upkara, set_uptkara, set_upfft

set_upkara([*threshold*])
set_uptkara([*threshold*])
set_upfft([*threshold*])

:: Set thresholds in the selection of an algorithm from N^2, Karatsuba, FFT algorithms for univariate polynomial multiplication.

return   value currently set

threshold   non-negative integer

- These functions set thresholds in the selection of an algorithm from N^2, Karatsuba, FFT algorithms for univariate polynomial multiplication.

- Products of univariate polynomials are computed by N^2, Karatsuba, FFT algorithms. The algorithm selection is done according to the degrees of input polynomials and the thresholds.
- See the description of each function for details.

References
            Section 6.4.2 [kmul ksquare ktmul], page 61, Section 6.4.1 [umul umul_ff usquare usquare_ff utmul utmul_ff], page 59.

6.4.4 utrunc, udecomp, ureverse

utrunc(*p*,*d*)
udecomp(*p*,*d*)
ureverse(*p*)
            :: Operations on polynomials

return   univariate polynomial or list of univariate polynomials

p   univariate polynomial

d   non-negative integer

- Let x be the variable of *p*. Then *p* can be decomposed as $p = p1+x^{(d+1)}p2$, where the degree of *p1* is less than or equal to *d*. Under the decomposition, utrunc() returns *p1* and udecomp() returns [*p1*,*p2*].
- Let *e* be the degree of *p* and *p*[*i*] the coefficient of *p* at degree *i*. Then ureverse() returns *p*[*e*]+*p*[*e*-1]x+....

    ```
    [132] utrunc((x+1)^10,5);
    252*x^5+210*x^4+120*x^3+45*x^2+10*x+1
    [133] udecomp((x+1)^10,5);
    [252*x^5+210*x^4+120*x^3+45*x^2+10*x+1,x^4+10*x^3+45*x^2+120*x+210]
    [134] ureverse(3*x^3+x^2+2*x);
    2*x^2+x+3
    ```

References
            Section 6.4.6 [udiv urem urembymul urembymul_precomp ugcd], page 63.

6.4.5 uinv_as_power_series, ureverse_inv_as_power_series

uinv_as_power_series(*p*,*d*)
ureverse_inv_as_power_series(*p*,*d*)
            :: Computes the truncated inverse as a power series.

return   univariate polynomial

p   univariate polynomial

d   non-negative integer

- For a polynomial *p* with a non zero constant term, uinv_as_power_series(*p*,*d*) computes a polynomial *r* whose degree is at most *d* such that $p*r = 1 \bmod x^{(d+1)}$, where x is the variable of *p*.
- Let *e* be the degree of *p*. ureverse_inv_as_power_series(*p*,*d*) computes uinv_as_power_series(*p1*,*d*) for *p1*=ureverse(*p*,*e*).

- The output of `ureverse_inv_as_power_series()` can be used as the input of
  `rembymul_precomp()`.

  ```
  [123] A=(x+1)^5;
  x^5+5*x^4+10*x^3+10*x^2+5*x+1
  [124] uinv_as_power_series(A,5);
  -126*x^5+70*x^4-35*x^3+15*x^2-5*x+1
  [126] A*R;
  -126*x^10-560*x^9-945*x^8-720*x^7-210*x^6+1
  [127] A=x^10+x^9;
  x^10+x^9
  [128] R=ureverse_inv_as_power_series(A,5);
  -x^5+x^4-x^3+x^2-x+1
  [129] ureverse(A)*R;
  -x^6+1
  ```

References

> Section 6.4.4 [utrunc udecomp ureverse], page 62, Section 6.4.6 [udiv urem
> urembymul urembymul_precomp ugcd], page 63.

6.4.6 `udiv`, `urem`, `urembymul`, `urembymul_precomp`, `ugcd`

`udiv(`*p1*`,`*p2*`)`
`urem(`*p1*`,`*p2*`)`
`urembymul(`*p1*`,`*p2*`)`
`urembymul_precomp(`*p1*`,`*p2*`,`*inv*`)`
`ugcd(`*p1*`,`*p2*`)`
> :: Division and GCD for univariate polynomials.

return   univariate polynomial

p1 p2 inv   univariate polynomial

- For univariate polynomials *p1* and *p2*, there exist polynomials *q* and *r* such that
  p1=*q**p2+*r* and the degree of *r* is less than that of *p2*. Then `udiv` returns *q*, `urem`
  and `urembymul` return *r*. `ugcd` returns the polynomial GCD of *p1* and *p2*. These
  functions are specially tuned up for dense univariate polynomials. In `urembymul` the
  division by *p2* is replaced with the inverse computation of *p2* as a power series and two
  polynomial multiplications. It speeds up the computation when the degrees of inputs
  are large.

- `urembymul_precomp` is efficient when one repeats divisions by a fixed polynomial. One
  has to compute the third argument by `ureverse_inv_as_power_series()`.

  ```
  [177] setmod_ff(2^160-47);
  1461501637330902918203684832716283019655932542929
  [178] A=randpoly_ff(200,x)$
  [179] B=randpoly_ff(101,x)$
  [180] cputime(1)$
  0sec(1.597e-05sec)
  [181] srem(A,B)$
  0.15sec + gc : 0.15sec(0.3035sec)
  ```

```
[182] urem(A,B)$
0.11sec + gc : 0.12sec(0.2347sec)
[183] urembymul(A,B)$
0.08sec + gc : 0.09sec(0.1651sec)
[184] R=ureverse_inv_as_power_series(B,101)$
0.04sec + gc : 0.03sec(0.063sec)
[185] urembymul_precomp(A,B,R)$
0.03sec(0.02501sec)
```

References

> Section 6.4.5 [uinv_as_power_series ureverse_inv_as_power_series], page 62.

6.5 Lists

6.5.1 car, cdr, cons, append, reverse, length

car(*list*)

> :: The first element of the given non-null list *list*.

cdr(*list*)

> :: A list obtained by removing the first element of the given non-null list *list*.

cons(*obj*,*list*)

> :: A list obtained by adding an element *obj* to the top of the given list *list*.

append(*list1*,*list2*)

> :: A list obtained by adding all elements in the list *list2* according to the order
> as it is to the last element in the list *list1*.

reverse(*list*)

> :: reversed list of *list*.

length(*list*|*vect*)

> :: Number of elements in a list *list* and a vector *vect*.

return   car() : arbitrary, cdr(), cons(), append(), reverse() : list, length() :
             non-negative integer

list list1 list2

> list

obj   arbitrary

- A list is written in **Asir** as [*obj1*,*obj2*,...]. Here, *obj1* is the first element.
- Function car() outputs the first element of a non-null list. For a null list, the result
  should be undefined. In the current implementation, however, it outputs a null list.
  This treatment for a null list may subject to change in future, and users are suggested
  not to use the tentative treatment for a null list for serious programming.
- Function cdr() outputs a list obtained by removing the first element from the input
  non-null list. For a null list, the result should be undefined. In the current implemen-
  tation, however, it outputs a null list. This treatment for a null list may subject to

change in future, and users are suggested not to use the tentative treatment for a null
list for serious programming.

- Function `cons()` composes a new list from the input list *list* and an arbitrary object *obj* by adding *obj* to the top of *list*.

- Function `append()` composes a new list, which has all elements of *list1* in the same ordering followed by all elements of *list2* in the same ordering.

- Function `reverse()` returns a reversed list of *list*.

- Function `length()` returns a non-negative integer which is the number of elements in the input list *list* and the input vector *vect*. Note that function `size` should be used for counting elements of *matrix*.

- Lists are read-only objects in **Asir**. There elements cannot be modified.

- The *n*-th element in a list can be referred to by applying the function `cdr()` *n* times repeatedly and `cdr()` at last. A more convenient way to access to the *n*-th element is the use of bracket notation, that is, to attach an index `[n]` like vectors and matrices. The system, however, follow the *n* pointers to access the desired element. Subsequently, much time is spent for an element located far from the top of the list.

- Function `cdr()` does not create a new cell (a memory quantity). Function `append()`, as a matter of fact, repeats `cons()` for as many as the length of *list1* the first argument. Subsequently, `append()` consumes much memory space if its first argument is long. Similar argument applies to function `reverse()`.

```
[0] L = [[1,2,3],4,[5,6]];
[[1,2,3],4,[5,6]]
[1] car(L);
[1,2,3]
[2] cdr(L);
[4,[5,6]]
[3] cons(x*y,L);
[y*x,[1,2,3],4,[5,6]]
[4] append([a,b,c],[d]);
[a,b,c,d]
[5] reverse([a,b,c,d]);
[d,c,b,a]
[6] length(L);
3
[7] length(ltov(L));
3
[8] L[2][0];
5
```

6.6 Arrays

6.6.1 `newvect, vector, vect`

`newvect(len[,list])`
`vector(len[,list])`

:: Creates a new vector object with its length *len*.

```
vect([elements])
```
           :: Creates a new vector object by *elements*.

return   vector

len   non-negative integer

list   list

elements   elements of the vector

- `vect` creates a new vector object by its elements.
- `vector` is an alias of `newvect`.
- `newvect` creates a new vector object with its length *len* and its elements all cleared to value 0. If the second argument, a list, is given, the vector is initialized by the list elements. Elements are used from the first through the last. If the list is short for initializing the full vector, 0's are filled in the remaining vector elements.
- Elements are indexed from 0 through *len*-1. Note that the first element has not index 1.
- List and vector are different types in **Asir**. Lists are conveniently used for representing many data objects whose size varies dynamically as computation proceeds. By its flexible expressive power, it is also conveniently used to describe initial values for other structured objects as you see for vectors. Access for an element of a list is performed by following pointers to next elements. By this, access costs for list elements differ for each element. In contrast to lists, vector elements can be accessed in a same time, because they are accessed by computing displacements from the top memory location of the vector object.

  Note also, in **Asir**, modification of an element of a vector causes modification of the whole vector itself, while modification of a list element does not cause the modification of the whole list object.

  By this, in **Asir** language, a vector element designator can be a left value of assignment statement, but a list element designator can NOT be a left value of assignment statement.
- No distinction of column vectors and row vectors in **Asir**. If a matrix is applied to a vector from left, the vector shall be taken as a column vector, and if from right it shall be taken as a row vector.
- The length (or size or dimension) of a vector is given by function `size()`.
- When a vector is passed to a function as its argument (actual parameter), the vector element can be modified in that function.
- A vector is displayed in a similar format as for a list. Note, however, there is a distinction: Elements of a vector are separated simply by a 'blank space', while those of a list by a 'comma.'
  ```
  [0] A=newvect(5);
  [ 0 0 0 0 0 ]
  [1] A=newvect(5,[1,2,3,4,[5,6]]);
  [ 1 2 3 4 [5,6] ]
  [2] A[0];
  ```

```
        1
    [3] A[4];
    [5,6]
    [4] size(A);
    [5]
    [5] length(A);
    5
    [6] vect(1,2,3,4,[5,6]);
    [ 1 2 3 4 [5,6] ]
    [7] def afo(V) { V[0] = x; }
    [8] afo(A)$
    [9] A;
    [ x 2 3 4 [5,6] ]
```

References
> Section 6.6.5 [newmat matrix], page 68, Section 6.6.7 [size], page 70, Section 6.6.2 [ltov], page 67, Section 6.6.3 [vtol], page 67.

6.6.2 `ltov`

`ltov(list)`
> :: Converts a list into a vector.

return   vector

list   list

- Converts a list *list* into a vector of same length. See also `newvect()`.

    ```
    [3] A=[1,2,3];
    [4] ltov(A);
    [ 1 2 3 ]
    ```

References
> Section 6.6.1 [newvect vector vect], page 66, Section 6.6.3 [vtol], page 67.

6.6.3 `vtol`

`vtol(vect)`
> :: Converts a vector into a list.

return   list

vect   vector

- Converts a vector *vect* of length n into a list [vect[0],...,vect[n-1]].
- A conversion from a list to a vector is done by `newvect()`.

    ```
    [3] A=newvect(3,[1,2,3]);
    [ 1 2 3 ]
    [4] vtol(A);
    [1,2,3]
    ```

References
> Section 6.6.1 [newvect vector vect], page 66, Section 6.6.2 [ltov], page 67.

6.6.4 `newbytearray`

`newbytearray(len,[listorstring])`
          :: Creates a new byte array.

return   byte array

len   non-negative integer

listorstring
          list or string

- This function generates a byte array. The specification is similar to that of `newvect`.
- The initial value can be specified by a character string.
- One can access elements of a byte array just as an array.

```
[182] A=newbytearray(3);
|00 00 00|
[183] A=newbytearray(3,[1,2,3]);
|01 02 03|
[184] A=newbytearray(3,"abc");
|61 62 63|
[185] A[0];
97
[186] A[1]=123;
123
[187] A;
|61 7b 63|
```

`References`
          Section 6.6.1 [`newvect vector vect`], page 66.

6.6.5 `newmat, matrix`

`newmat(row,col [,[[a,b,...],[c,d,...],...]])`
`matrix(row,col [,[[a,b,...],[c,d,...],...]])`
          :: Creates a new matrix with *row* rows and *col* columns.

return   matrix

row col   non-negative integer

a b c d   arbitrary

- `matrix` is an alias of `newmat`.
- If the third argument, a list, is given, the newly created matrix is initialized so that each element of the list (again a list) initializes each of the rows of the matrix. Elements are used from the first through the last. If the list is short, 0's are filled in the remaining matrix elements. If no third argument is given all the elements are cleared to 0.
- The size of a matrix is given by function `size()`.
- Let `M` be a program variable assigned to a matrix. Then, `M[I]` denotes a (row) vector which corresponds with the `I`-th row of the matrix. Note that the vector shares its element with the original matrix. Subsequently, if an element of the vector is modified, then the corresponding matrix element is also modified.

- When a matrix is passed to a function as its argument (actual parameter), the matrix element can be modified within that function.

```
[0] A = newmat(3,3,[[1,1,1],[x,y],[x^2]]);
[ 1 1 1 ]
[ x y 0 ]
[ x^2 0 0 ]
[1] det(A);
-y*x^2
[2] size(A);
[3,3]
[3] A[1];
[ x y 0 ]
[4] A[1][3];
getarray : Out of range
return to toplevel
```

References
          Section 6.6.1 [newvect vector vect], page 66, Section 6.6.7 [size], page 70,
          Section 6.6.8 [det nd_det invmat], page 70.

6.6.6 mat, matr, matc

mat(*vector*[,...])
matr(*vector*[,...])
          :: Creates a new matrix by list of row vectors.

matc(*vector*[,...])
          :: Creates a new matrix by list of column vectors.

return   matrix

vector   array or list

- mat is an alias of matr.

- Each vector has same length. Elements are used from the first through the last. If the list is short, 0's are filled in the remaining matrix elements.

```
[0] matr([1,2,3],[4,5,6],[7,8]);
[ 1 2 3 ]
[ 4 5 6 ]
[ 7 8 0 ]
[1] matc([1,2,3],[4,5,6],[7,8]);
[ 1 4 7 ]
[ 2 5 8 ]
[ 3 6 0 ]
```

References
          Section 6.6.5 [newmat matrix], page 68

6.6.7 `size`

`size(`*vect*`|`*mat*`)`

:: A list containing the number of elements of the given vector, [`size of `*vect*`]`, or a list containing row size and column size of the given matrix, [`row size of `*mat*`, column size of `*mat*`]`.

return   list

vect   vector

mat   matrix

- Return a list consisting of the dimension of the vector *vect*, or a list consisting of the row size and column size of the matrix *matrix*.
- Use `length()` for the size of *list*, and `nmono()` for the number of monomials with non-zero coefficients in a rational expression.

```
[0] A = newvect(4);
[ 0 0 0 0 ]
[1] size(A);
[4]
[2] length(A);
4
[3] B = newmat(2,3,[[1,2,3],[4,5,6]]);
[ 1 2 3 ]
[ 4 5 6 ]
[4] size(B);
[2,3]
```

`References`

Section 6.5.1 [`car cdr cons append reverse length`], page 64, Section 6.3.6 [`nmono`], page 49.

6.6.8 `det, nd_det, invmat`

`det(`*mat*`[,`*mod*`])`
`nd_det(`*mat*`[,`*mod*`])`

:: Determinant of *mat*.

`invmat(`*mat*`)`

:: Inverse matrix of *mat*.

return   det: expression, `invmat`: list

mat   matrix

mod   prime

- `det` and `nd_det` compute the determinant of matrix *mat*. `invmat` computes the inverse matrix of matrix *mat*. `invmat` returns a list [`num,den`], where `num` is a matrix and `num/den` represents the inverse matrix.
- The computation is done over GF(*mod*) if *mod* is specitied.

- The fraction free Gaussian algorithm is employed. For matrices with multi-variate polynomial entries, minor expansion algorithm sometimes is more efficient than the fraction free Gaussian algorithm.

- `nd_det` can be used for computing the determinant of a matrix with polynomial entries over the rationals or finite fields. The algorithm is an improved version of the fraction free Gaussian algorithm and it computes the determinant faster than `det`.

```
[91] A=newmat(5,5)$
[92] V=[x,y,z,u,v];
[x,y,z,u,v]
[93] for(I=0;I<5;I++)for(J=0,B=A[I],W=V[I];J<5;J++)B[J]=W^J;
[94] A;
[ 1 x x^2 x^3 x^4 ]
[ 1 y y^2 y^3 y^4 ]
[ 1 z z^2 z^3 z^4 ]
[ 1 u u^2 u^3 u^4 ]
[ 1 v v^2 v^3 v^4 ]
[95] fctr(det(A));
[[1,1],[u-v,1],[-z+v,1],[-z+u,1],[-y+u,1],[y-v,1],[-y+z,1],[-x+u,1],
[-x+z,1],[-x+v,1],[-x+y,1]]
[96] A = newmat(3,3)$
[97] for(I=0;I<3;I++)for(J=0,B=A[I],W=V[I];J<3;J++)B[J]=W^J;
[98] A;
[ 1 x x^2 ]
[ 1 y y^2 ]
[ 1 z z^2 ]
[99] invmat(A);
[[ -z*y^2+z^2*y z*x^2-z^2*x -y*x^2+y^2*x ]
[ y^2-z^2 -x^2+z^2 x^2-y^2 ]
[ -y+z x-z -x+y ],(-y+z)*x^2+(y^2-z^2)*x-z*y^2+z^2*y]
[100] A*B[0];
[ (-y+z)*x^2+(y^2-z^2)*x-z*y^2+z^2*y 0 0 ]
[ 0 (-y+z)*x^2+(y^2-z^2)*x-z*y^2+z^2*y 0 ]
[ 0 0 (-y+z)*x^2+(y^2-z^2)*x-z*y^2+z^2*y ]
[101] map(red,A*B[0]/B[1]);
[ 1 0 0 ]
[ 0 1 0 ]
[ 0 0 1 ]
```

References
       Section 6.6.5 [`newmat matrix`], page 68.

6.6.9 `qsort`

`qsort(array[,func])`
       :: Sorts an array *array*.

return   *array* (The same as the input; Only the elements are exchanged.)

array   array

func   function for comparison
- This function sorts an array by *quick sort*.
- If *func* is not specified, the built-in comparison function is used and the array is sorted in increasing order.
- If a function of two arguments *func* which returns 0, 1, or -1 is provided, then an ordering is determined so that `A<B` if `func(A,B)=1` holds, and the array is sorted in increasing order with respect to the ordering.
- The returned array is the same as the input. Only the elements are exchanged.

```
[0] qsort(newvect(10,[1,4,6,7,3,2,9,6,0,-1]));
[ -1 0 1 2 3 4 6 6 7 9 ]
[1] def rev(A,B) { return A>B?-1:(A<B?1:0); }
[2] qsort(newvect(10,[1,4,6,7,3,2,9,6,0,-1]),rev);
[ 9 7 6 6 4 3 2 1 0 -1 ]
```

`References`
        Section 6.3.7 [`ord`], page 49, Section 6.3.2 [`vars`], page 47.

6.6.10 `rowx, rowm, rowa, colx, colm, cola`

`rowx(matrix,i,j)`
        :: Exchanges the *i*-th and *j*-th rows.

`rowm(matrix,i,c)`
        :: Multiplies the *i*-th row by *c*.

`rowa(matrix,i,c)`
        :: Appends *c* times the *j*-th row to the *j*-th row.

`colx(matrix,i,j)`
        :: Exchanges the *i*-th and *j*-th columns.

`colm(matrix,i,c)`
        :: Multiplies the *i*-th column by *c*.

`cola(matrix,i,c)`
        :: Appends *c* times the *j*-th column to the *j*-th column.

return   matrix

i, j   integers

c   coefficient

- These operations are destructive for the matrix.

```
[0] A=newmat(3,3,[[1,2,3],[4,5,6],[7,8,9]]);
[ 1 2 3 ]
[ 4 5 6 ]
[ 7 8 9 ]
[1] rowx(A,1,2)$
[2] A;
[ 1 2 3 ]
[ 7 8 9 ]
```

```
[ 4 5 6 ]
[3] rowm(A,2,x);
[ 1 2 3 ]
[ 7 8 9 ]
[ 4*x 5*x 6*x ]
[4] rowa(A,0,1,z);
[ 7*z+1 8*z+2 9*z+3 ]
[ 7 8 9 ]
[ 4*x 5*x 6*x ]
```

References
> Section 6.6.5 [`newmat matrix`], page 68

6.7 Structures

6.7.1 `newstruct`

`newstruct(`*name*`)`
> :: Creates a new structure object whose name is *name*.

return   structure

name   string

- This function creates an new structure object whose name is *name*.

- A structure named *name* should be defined in advance.

- Each member of a structure is specified by its name using the operator `->`. If the specified member is also an structure, the specification by `->` can be nested.

  ```
  [0] struct list {h,t};
  0
  [1] A=newstruct(list);
  {0,0}
  [2] A->t = newstruct(list);
  {0,0}
  [3] A;
  {0,{0,0}}
  [4] A->h = 1;
  1
  [5] A->t->h = 2;
  2
  [6] A->t->t = 3;
  3
  [7] A;
  {1,{2,3}}
  ```

References
> Section 6.7.2 [`arfreg`], page 74, Section 4.2.9 [`structure definition`], page 25

6.7.2 `arfreg`

`arfreg(`*name*`,`*add*`,`*sub*`,`*mul*`,`*div*`,`*pwr*`,`*chsgn*`,`*comp*`)`
           :: Registers a set of fundamental operations for a type of structure.

return   1

name   string

add sub mul div pwr chsgn comp
           user defined functions

- This function registers a set of fundamental operations for a type of structure whose name is *name*.

- The specification of each function is as follows.

  `add(A,B)`   A+B

  `sub(A,B)`   A-B

  `mul(A,B)`   A*B

  `div(A,B)`   A/B

  `pwr(A,B)`   A^B

  `chsgn(A)`   -A

  `comp(A,B)`
           1,0,-1 according to the result of a comparison between A and B.

```
% cat test
struct a {id,body}$

def add(A,B)
{
  C = newstruct(a);
  C->id = A->id; C->body = A->body+B->body;
  return C;
}

def sub(A,B)
{
  C = newstruct(a);
  C->id = A->id; C->body = A->body-B->body;
  return C;
}

def mul(A,B)
{
  C = newstruct(a);
  C->id = A->id; C->body = A->body*B->body;
  return C;
}
```

```
def div(A,B)
{
  C = newstruct(a);
  C->id = A->id; C->body = A->body/B->body;
  return C;
}

def pwr(A,B)
{
  C = newstruct(a);
  C->id = A->id; C->body = A->body^B;
  return C;
}

def chsgn(A)
{
  C = newstruct(a);
  C->id = A->id; C->body = -A->body;
  return C;
}

def comp(A,B)
{
  if ( A->body > B->body )
    return 1;
  else if ( A->body < B->body )
    return -1;
  else
    return 0;
}

arfreg("a",add,sub,mul,div,pwr,chsgn,comp)$
end$
% asir
This is Risa/Asir, Version 20000908.
Copyright (C) FUJITSU LABORATORIES LIMITED.
1994-2000. All rights reserved.
[0] load("./test")$
[11] A=newstruct(a);
{0,0}
[12] B=newstruct(a);
{0,0}
[13] A->body = 3;
3
[14] B->body = 4;
4
```

```
[15] A*B;
{0,12}
```

References
Section 6.7.1 [newstruct], page 73, Section 4.2.9 [structure definition], page 25

6.7.3 struct_type

struct_type(*name*|*object*)
:: Get an identity number of the structure of *object* and *name*.

return   an integer

name   string

object   a structure

- struct_type() returns an identity number of the structure or -1 (if an error occurs).

```
[10] struct list {h,t};
0
[11] A=newstruct(list);
{0,0}
[12] struct_type(A);
3
[13] struct_type("list");
3
```

References
Section 6.7.1 [newstruct], page 73, Section 4.2.9 [structure definition], page 25

6.8 Types

6.8.1 type

type(*obj*)
:: Returns an integer which identifies the type of the object *obj* in question.

return   integer

obj   arbitrary

- Current assignment of integers for object types is listed below.

0	0
1	number
2	polynomial (not number)
3	rational expression (not polynomial)
4	list
5	vector

6	matrix
7	string
8	structure
9	distributed polynomial
10	32bit unsigned integer
11	error object
12	matrix over GF(2)
13	MATHCAP object
14	first order formula
-1	VOID object

- For further classification of *number*, use `ntype()`. For further classification of *variable*, use `vtype()`.

References
> Section 6.8.2 [`ntype`], page 77, Section 6.8.3 [`vtype`], page 78.

6.8.2 `ntype`

`ntype(num)`
> :: Classifier of type *num*. Returns a sub-type number, an integer, for *obj*.

return   integer

obj   number

- Sub-types for type number are listed below.

0	rational number
1	floating double (double precision floating point number)
2	algebraic number over rational number field
3	arbitrary precision floating point number (**bigfloat**)
4	complex number
5	element of a finite field
6	element of a large finite prime field
7	element of a finite field of characteristic 2

- When arithmetic operations for numbers are performed, type coercion will be taken if their number sub-types are different so that the object having smaller sub-type number will be transformed to match the other object, except for algebraic numbers.
- A number object created by `newalg(x^2+1)` and the unit of imaginary number `@i` have different number sub-types, and it is treated independently.
- See Chapter 9 [Algebraic numbers], page 162, for algebraic numbers.

```
[0]  [10/37,ntype(10/37)];
[10/37,0]
[1]  [10.0/37.0,ntype(10.0/37.0)];
[0.27027,1]
[2]  [newalg(x^2+1)+1,ntype(newalg(x^2+1)+1)];
[(#0+1),2]
[3]  [eval(sin(@pi/6)),ntype(eval(sin(@pi/6)))];
[0.49999999999999999991,3]
[4]  [@i+1,ntype(@i+1)];
[(1+1*@i),4]
```

References
          Section 6.8.1 [type], page 76.

6.8.3 `vtype`

`vtype(var)`
          :: Type of indetarminates *var*.

return   integer

var   indeterminate

- Classify indeterminates into sub-types by giving an integer value as follows. For details See Section 3.3 [Types of indeterminates], page 16.

    0   ordinary indeterminate, which can be directly typed in on a keyboard (a,b,x,afo,bfo,...,etc.)

    1   Special indeterminate, created by `uc()` (_0, _1, _2, ... etc.)

    2   function form (`sin(x)`, `log(a+1)`, `acosh(1)`, `@pi`, `@e`, ... etc.)

    3   functor (built-in functor name, user defined functor, functor for the elementary functions) : `sin`, `log`, ... etc)

- Note: An input 'a();' will cause an error, but it changes the system database for identifiers. After this error, you will find 'vtype(a)' will result 3. (Identifier `a` is registered as a user defined functor).

- Usually `@pi` and `@e` are treated as indeterminates, whereas they are treated as numbers within functions `eval()` and `pari()`.

References
          Section 6.8.1 [type], page 76, Section 6.8.2 [ntype], page 77, Section 6.3.3 [uc], page 47.

6.9 Operations on functions

6.9.1 `call`

`call(name, args)`
          :: Call the function *name* with *args*.

return   a return value of `name()`.

name   indefinite (function name)

args   a list of arguments

- See Section 6.8.3 [vtype], page 78, for function form.

```
[0] A=igcd;
igcd
[1] call(A,[4,6]);
2
[2] (*A)(4,6);
2
```

References
          Section 6.8.3 [vtype], page 78.

6.9.2 `functor, args, funargs`

`functor(`*func*`)`
          :: Functor of function form *func*.

`args(`*func*`)`
          :: List of arguments of function form *func*.

`funargs(`*func*`)`
          :: `cons(functor(`*func*`),args(`*func*`))`.

return   `functor()` : indeterminate, `args()`, `funargs()` : list

func   function form

- See Section 6.8.3 [vtype], page 78, for function form.

- Extract the functor and the arguments of function form *func*.

- Assign a program variable, say `F`, to the functor obtained by `functor()`. Then, you can type `(*F)(x)` (, or `(*F)(x,y,...)` depending on the arity,) to input a function form with argument x.

```
[0] functor(sin(x));
sin
[0] args(sin(x));
[x]
[0] funargs(sin(3*cos(y)));
[sin,3*cos(y)]
[1] for (L=[sin,cos,tan];L!=[];L=cdr(L)) {A=car(L);
print(eval((*A)(@pi/3)));}
0.86602540349122136831
0.5000000002
1.7320508058
```

References
          Section 6.8.3 [vtype], page 78.

6.10 Strings

6.10.1 `rtostr`

`rtostr(`*obj*`)`

:: Convert *obj* into a string.

return   string

obj   arbitrary

- Convert an arbitrary object *obj* into a string.
- This function is convenient to create variables with numbered (or indexed) names by converting integers into strings and appending them to some name strings.
- Use `strtov()` for inverse conversion from string to indeterminate.

  ```
  [0] A=afo;
  afo
  [1] type(A);
  2
  [2] B=rtostr(A);
  afo
  [3] type(B);
  7
  [4] B+"1";
  afo1
  ```

`References`

Section 6.10.2 [`strtov`], page 80, Section 6.8.1 [`type`], page 76.

6.10.2 `strtov`

`strtov(`*str*`)`

:: Convert a string *str* into an indeterminate.

return   intedeterminate

str   string which is valid to constitute an indeterminate.

- Convert a string that is valid for an indeterminate into an indeterminate which have *str* as its print name.
- The valid string for an indeterminate is such a string that begins with a small alphabetical letter possibly followed by any string composed of alphabetical letters, digits or a symbol '`_`'.
- Use the command to create indeterminates dynamically in programs.

  ```
  [0] A="afo";
  afo
  [1] for (I=0;I<3;I++) {B=strtov(A+rtostr(I)); print([B,type(B)]);}
  [afo0,2]
  [afo1,2]
  [afo2,2]
  ```

References

Section 6.10.1 [rtostr], page 80, Section 6.8.1 [type], page 76, Section 6.3.3 [uc], page 47.

6.10.3 eval_str

eval_str(*str*)
:: Evaluates a string *str*.

return   object

str   string which can be accepted by **Asir** parser

- This function evaluates a string which can be accepted by **Asir** parser and returns the result.

- The input string should represent an expression.

- This functions is the inversion function of rtostr().

```
[0] eval_str("1+2");
3
[1] fctr(eval_str(rtostr((x+y)^10)));
[[1,1],[x+y,10]]
```

References

Section 6.10.1 [rtostr], page 80

6.10.4 strtoascii, asciitostr

strtoascii(*str*)
:: Converts a string into a sequence of ASCII codes.

asciitostr(*list*)
:: Converts a sequence of ASCII codes into a string.

return   strtoascii():list; asciitostr():string

str   string

list   list containing positive integers less than 256.

- strtoascii() converts a string into a list of integers which is a representation of the string by the ASCII code.

- asciitostr() is the inverse of asciitostr().

```
[0] strtoascii("abcxyz");
[97,98,99,120,121,122]
[1] asciitostr(@);
abcxyz
[2] asciitostr([256]);
asciitostr : argument out of range
return to toplevel
```

6.10.5 `str_len, str_chr, sub_str`

`str_len(`*str*`)`
>> :: Returns the length of a string.

`str_chr(`*str*`,`*start*`,`*c*`)`
>> :: Returns the position of the first occurrence of a character in a string.

`sub_str(`*str*`,`*start*`,`*end*`)`
>> :: Returns a substring of a string.

return   `str_len()`, `str_chr()`:integer; `sub_str()`:string

str c   string

start end   non-negative integer

- `str_len()` returns the length of a string.
- `str_chr()` scans a string *str* from the *start*-th character and returns the position of the first occurrence of the first character of a string *c*. Note that the top of a string is the 0-th character. It returns -1 if the character does not appear.
- `sub_str()` generates a substring of *str* containing characters from the *start*-th one to the *end*-th one.

```
[185] Line="123 456 (x+y)^3";
123 456 (x+y)^3
[186] Sp1 = str_chr(Line,0," ");
3
[187] D0 = eval_str(sub_str(Line,0,Sp1-1));
123
[188] Sp2 = str_chr(Line,Sp1+1," ");
7
[189] D1 = eval_str(sub_str(Line,Sp1+1,Sp2-1));
456
[190] C = eval_str(sub_str(Line,Sp2+1,str_len(Line)-1));
x^3+3*y*x^2+3*y^2*x+y^3
```

6.11 Inputs and Outputs

6.11.1 `end, quit`

`end, quit`
>> :: Close the currently reading file. At the top level, terminate the **Asir** session.

- These two functions take no arguments. These functions can be called without a '()'. Either function close the current input file. This means the termination of the **Asir** session at the top level.
- An input file will be automatically closed if it is read to its end. However, if no `end$` is written at the last of the input file, the control will be returned to the top level and **Asir** will be waiting for an input without any prompting. Thus, in order to avoid confusion, putting a `end$` at the last line of the input file is strongly recommended.

```
      [6] quit;
      %
```

References

> Section 6.11.2 [`load`], page 83.

6.11.2 `load`

`load("`*filename*`")`

> :: Reads a program file *filename*.

return   (1|0)

filename   file (path) name

- See Chapter 4 [User language Asir], page 18, for practical programming. Since text files are read through `cpp`, the user can use, as in C programs, `#include` and `#define` in **Asir** program source codes. The `cpp` which is installed to the system with a C compler is used in the unix version. The `mcpp` `http://mcpp.sourceforge.net` is used in the Windows version. Note that the length of a line has a limit for an input `mcpp`. The `OpenXM/bin/ox_cpp` is used in the unix/OpenXM version (including cfep/asir for MacOS X). This is the `cpp` distributed with the Portable C compiler `http://pcc.ludd.ltu.se`.

- It returns 1 if the designated file exists, 0 otherwise.

- If the *filename* begins with '`/`', it is understood as an absolute path name; with '`.`', relative path name from current directory; otherwise, the file is searched first from directories assigned to an environmental variable `ASIRLOADPATH`, then if the search ends up in failure, the standard library directory (or directories assigned to `ASIR_LIBDIR`) shall be searched. On Windows, `get_rootdir()/lib` is searched if `ASIR_LIBDIR` is not set.

- We recommend to write an `end` command at the last line of your program. If not, **Asir** will not give you a prompt after it will have executed `load` command. (Escape with an interrupt character (Section 2.7 [Interruption], page 8), if you have lost yourself.) Even in such a situation, **Asir** itself is still ready to read keyboard inputs as usual. It is, however, embarrassing and may cause other errors. Therefore, to put an `end$` at the last line is desirable. (Command `end;` will work as well, but it also returns and displays verbose.)

- On Windows one has to use '`/`' as the separator of directory names.

References

> Section 6.11.1 [`end quit`], page 82, Section 6.11.3 [`which`], page 83, Section 6.14.16 [`get_rootdir`], page 101.

6.11.3 `which`

`which("`*filename*`")`

> :: This returns the path name for the *filename* which `load()` will read.

return   path name

filename   filename (path name) or 0

- This function searches directory trees according to the same procedure as `load()` will do. Then, returns a string, the path name to the file if the named file exists; 0 unless otherwise.
- For details of searching procedure, refer to the description about `load()`.
- On Windows one has to use '/' as the separator of directory names.

```
[0] which("gr");
./gb/gr
[1] which("/usr/local/lib/gr");
0
[2] which("/usr/local/lib/asir/gr");
/usr/local/lib/asir/gr
```

References
   Section 6.11.2 [`load`], page 83.

6.11.4 `output`

`output(["filename"])`
   :: Writes the return values and prompt onto file *filename*.

return   1

filename   filename

- Standard output stream of **Asir** is redirected to the specified file. While **Asir** is writing its outputs onto a file, no outputs, except for keyboard inputs and some of error messages, are written onto the standard output. (You cannot see the result on the display.)
- To direct the **Asir** outputs to the standard output, issue the command without argument, i.e., `output()`.
- If the specified file already exists, new outputs will be added to the tail of the file. If not, a file is newly created and the outputs will be written onto the file.
- When file name is specified without double quotes (`""`), or when protected file is specified, an error occurs and the system returns to the top level.
- If you want to write inputs from the key board onto the file as well as **Asir** outputs, put command `ctrl("echo",1)`, and then redirect the standard output to your desired file.
- Contents which are written onto the standard error output, CPU time etc., are not written onto the file.
- Reading and writing algebraic expressions which contain neither functional forms nor unknown coefficients (`vtype()` References) are performed more efficiently, with respect to both time and space, by `bload()` and `bsave()`.
- On Windows one has to use '/' as the separator of directory names.

```
[83] output("afo");
fctr(x^2-y^2);
print("afo");
output();
1
```

```
     [87] quit;
     % cat afo
     1
     [84] [[1,1],[x+y,1],[x-y,1]]
     [85] afo
     0
     [86]
```

References
          Section 6.14.1 [ctrl], page 91, Section 6.11.5 [bsave bload], page 85.

6.11.5 bsave, bload

bsave(*obj*,"*filename*")
          :: This function writes *obj* onto *filename* in binary form.

bload("*filename*")
          :: This function reads an expression from *filename* in binary form.

return   bsave() : 1, bload() : the expression read

obj   arbitrary expression which does not contain neither function forms nor unknown
               coefficients.

filename   filename

- Function bsave() writes an object onto a file in its internal form (not exact internal
  form but very similar). Function bload() read the expression from files which is written
  by bsave(). Current implementation support arbitrary expressions, including lists,
  arrays (i.e., vectors and matrices), except for function forms and unknown coefficients
  (vtype() References.)

- The parser is activated to retrieve expressions written by output() , whereas internal
  forms are directly reconstructed by bload() from the bsave()'ed object in the file.
  The latter is much more efficient with respect to both time and space.

- It may happen that the variable ordering at reading is changed from that at writing. In
  such a case, the variable ordering in the internal expression is automatically rearranged
  according to the current variable ordering.

- On Windows one has to use '/' as the separator of directory names.

  ```
  [0] A=(x+y+z+u+v+w)^20$
  [1] bsave(A,"afo");
  1
  [2] B = bload("afo")$
  [3] A == B;
  1
  [4] X=(x+y)^2;
  x^2+2*y*x+y^2
  [5] bsave(X,"afo")$
  [6] quit;
  % asir
  [0] ord([y,x])$
  ```

```
[1] bload("afo");
y^2+2*x*y+x^2
```

```
References
```
        Section 6.11.4 [output], page 84.

6.11.6 bload27

```
bload27("filename")
```
        :: Reads bsaved file created by older version of **Asir**.

return   expression read

filename   filename

- In older versions an arbitrary precision integer is represented as an array of 27bit integers. In the current version it is represented as an array of 32bit integers. By this incompatibility the bsaved file created by older versions cannot be read in the current version by bload. bload27 is used to read such files.

- On Windows one has to use '/' as the separator of directory names.

```
References
```
        Section 6.11.5 [bsave bload], page 85.

6.11.7 print

```
print(obj [,nl])
```
        :: Displays (or outputs) *obj*.

return   0

obj   arbitrary

nl   flag (arbitrary)

- Displays (or outputs) *obj*.

- It normally adds linefeed code to cause the cursor moving to the next line. If 0 or 2 is given as the second argument, it does not add a linefeed. If the second argument is 0, the output is simply written in the buffer. If the second argument is 2, the output is flushed.

- The return value of this function is 0. If command print(rat); is performed at the top level, first the value of *rat* will be printed, followed by a linefeed, followed by a 0 which is the value of the function and followed by a linefeed and the next prompt. (If the command is terminated by a '$', e.g., print(rat)$, The last 0 will not be printed.)

- Formatted outputs are not currently supported. If one wishes to output multiple objects by a single print() command, use list like [obj1,...], which is not so beautiful, but convenient to minimize programming efforts.

```
[8] def cat(L) { while ( L != [] ) { print(car(L),0); L = cdr(L);}
print(""); }
[9] cat([xyz,123,"gahaha"])$
xyz123gahaha
```

6.11.8 `access`

`access(file)`

   :: testing an existence of *file*.

return (1|0)

file   filename

6.11.9 `remove_file`

`remove_file(file)`

   :: Delete an file *file*.

return 1

file   filename

6.11.10 `open_file`, `close_file`, `get_line`, `get_byte`, `put_byte`, `purge_` `stdin`

`open_file("filename"[,"mode"])`

   :: Opens *filename* for reading.

`close_file(num)`

   :: Closes the file indicated by a descriptor *num*.

`get_line([num])`

   :: Reads a line from the file indicated by a descriptor *num*.

`get_byte(num)`

   :: Reads a byte from the file indicated by a descriptor *num*.

`put_byte(num,c)`

   :: Writes a byte *c* to the file indicated by a descriptor *num*.

`purge_stdin()`

   :: Clears the buffer for the standard input.

return `open_file()` : integer (fild id); `close_file()` : 1; `get_line()` : string; `get_` `byte()`, `put_byte()` : integer

filename file (path) name

mode string

num non-negative integer (file descriptor)

- `open_file()` opens a file. If *mode* is not specified, a file is opened for reading. If *mode* is specified, it is used as the mode specification for C standard I/O function `fopen()`. For example `"w"` requests that the file is truncated to zero length or created for writing. `"a"` requests that the file is opened for writing or created if it does not exist. The stream pointer is set at the end of the file. If successful, it returns a non-negative integer as the file descriptor. Otherwise the system error function is called. Unnecessary files should be closed by `close_file()`. If the special file name unix://stdin or unix://stdout or unix://stderr is given, it returns the file descriptor for the standard input or the standard output or the standard error stream respectively. The mode argument is ignored in this case.

- `get_line()` reads a line from an opened file and returns the line as a string. If no argument is supplied, it reads a line from the standard input.
- `get_byte()` reads a byte from an opened file and returns the it as an integer.
- `put_byte()` writes a byte from an opened file and returns the the byte as an integer.
- A `get_line()` call after reading the end of file returns an integer 0.
- Strings can be converted into internal forms with string manipulation functions such as `sub_str()`, `eval_str()`.
- `purge_stdin()` clears the buffer for the standard input. When a function receives a character string from `get_line()`, this functions should be called in advance in order to avoid an incorrect behavior which is caused by the characters already exists in the buffer.

```
[185] Id = open_file("test");
0
[186] get_line(Id);
12345

[187] get_line(Id);
67890

[188] get_line(Id);
0
[189] type(@@);
0
[190] close_file(Id);
1
[191] open_file("test");
1
[192] get_line(1);
12345

[193] get_byte(1);
54                    /* the ASCII code of '6' */
[194] get_line(1);
7890                  /* the rest of the last line */
[195] def test() { return get_line(); }
[196] def test1() { purge_stdin(); return get_line(); }
[197] test();
                      /* a remaining newline character has been read */
                      /* returns immediately */
[198] test1();
123;                  /* input from a keyboard */
123;                  /* returned value */

[199]
```

References

> Section 6.10.3 [`eval_str`], page 81, Section 6.10.5 [`str_len str_chr sub_str`], page 82.

6.12 Operations for modules

6.12.1 `module_list`

`module_list()`

> :: Get the list of loaded modules.

return   The list of loaded modules.

```
[1040] module_list();
[gr,primdec,bfct,sm1,gnuplot,tigers,phc]
```

References

> See Section 4.2.13 [module], page 28.

6.12.2 `module_definedp`

`module_definedp(name)`

> :: Testing an existence of the module *name*.

return   (1|0)

name   a module name

- If the module *name* exists, then `module_definedp` returns 1. otherwise 0.

```
[100] module_definedp("gr");
1
```

References

> Section 6.12.1 [`module_list`], page 89, See Section 4.2.13 [module], page 28.

6.12.3 `remove_module`

`remove_module(name)`

> :: Remove the module *name*.

return   (1|0)

name   a module name

-

```
[100] remove_module("gr");
1
```

References

> See Section 4.2.13 [module], page 28.

6.13 Numerical functions

6.13.1 `dacos, dasin, datan, dcos, dsin, dtan`

`dacos(num)`
:: Get the value of Arccos of *num*.

`dasin(num)`
:: Get the value of Arcsin of *num*.

`datan(num)`
:: Get the value of Arctan of *num*.

`dcos(num)`
:: Get the value of cos of *num*.

`dsin(num)`
:: Get the value of sin of *num*.

`dtan(num)`
:: Get the value of tan of *num*.

return   floating point number

num   number

- Compute numerical values of trigonometric functions.
- These functions use the standard mathematical library of C language. So results depend on operating systems and a C compilers.

```
[0] 4*datan(1);
3.14159
```

6.13.2 `dabs, dexp, dlog, dsqrt`

`dabs(num)`
:: Get the absolute value of *num*.

`dexp(num)`
:: Get the value of expornent of *num*.

`dlog(num)`
:: Get the value of logarithm of *num*.

`dsqrt(num)`
:: Get the value of square root of *num*.

return   floating point number

num   number

- Compute numerical values of elementary functions.
- These functions use the standard mathematical library of C language. So results depend on operating systems and a C compilers.

```
[0] dexp(1);
2.71828
```

6.13.3 `ceil`, `floor`, `rint`, `dceil`, `dfloor`, `drint`

`ceil(`*num*`)`
`dceil(`*num*`)`
>   :: Get the ceiling integer of *num*.

`floor(`*num*`)`
`dfloor(`*num*`)`
>   :: Get the floor integer of *num*.

`rint(`*num*`)`
`drint(`*num*`)`
>   :: Get the round integer of *num*.

return   integer

num   number
```
    [0] dceil(1.1);
    1
```

6.14 Miscellaneouses

6.14.1 `ctrl`

`ctrl("`*switch*`"[,`*obj*`])`
>   :: Sets the value of *switch*.

`ctrl()`   :: Show the values of *switch*'s.

`ctrl(|desc=1)`
>   :: Show the values of *switch*'s and short descriptions

return   value of *switch*

switch   switch name

obj   parameter

- This function is used to set or to get the values of switches. The switches are used to control an execution of **Asir**.
- If *obj* is not specified, the value of *switch* is returned.
- If *obj* is specified, the value of *switch* is set to *obj*.
- Switches are specified by strings, namely, enclosed by two double quotes.
- Here are of switches of **Asir**.

    cputime   If 'on', CPU time and GC time is displayed at every top level evaluation of **Asir** command; if 'off', not displayed. See Section 6.14.6 [cputime tstart tstop], page 96. (The switch is also set by command `cputime(1)`, and reset by `cputime(0)`.)

    nez   Selection for EZGCD algorithm. It is set to 1 by default. Ordinary users need not change this setting.

echo
:   If 'on', inputs from the standard input will be echoed onto the standard output. When executing to load a file, the contents of the file will be written onto the standard output. If 'off', the inputs will not be echoed. This command will be useful when used with command `output`.

bigfloat
:   If 'on', floating operations will be done by **PARI** system with arbitrary precision floating point operations. Default precision is set to 9 digits. To change the precision, use command `setbprec setprec`. If 'off', floating operations will be done by **Asir**'s own floating operation routines with a fixed precision operations of standard floating double.

adj
:   Sets the frequency of garbage collection. A rational number greater than or equal to 1 can be specified. The default value is 3. If a value closer to 1 is specified, larger heap is allocated and as a result, the frequency of garbage collection decreases. See Section 2.4 [Command line options], page 6.

verbose
:   If 'on' a warning messages is displayed when a function is redefined.

quiet_mode
:   If 1 is set, the copyright notice has been displayed at boot time.

prompt
:   If the value is 0, then prompt is not output. If the value is 1, then the standard prompt is output. Asir prompt can be customized by giving a C-style format string. Example (for unix asir); `ctrl("prompt","\033[32m[%d]:= \033[0m")`

hex
:   If 1 is set, integers are displayed as hexadecimal numbers with prefix `0x`. if -1 is set, hexadecimal numbers are displayed with '|' inserted at every 8 hexadecimal digits.

real_digit
:   Sets the number of digits used to print a floating double.

double_output
:   If set to 1, any floating double is printed in the style ddd.ddd.

fortran_output
:   If 'on' polynomials are displayed in FORTRAN style. That is, a power is represented by '`**`' instead of '`^`'. The default value is 'off.

ox_batch
:   If 'on', the OpenXM send buffer is flushed only when the buffer is full. If 'off', the buffer is always flushed at each sending of data or command. The default value is 'off'. See Chapter 7 [Distributed computation], page 102.

ox_check
:   If 'on' the check by mathcap is done before sending data. The default value is 'on'. See Chapter 7 [Distributed computation], page 102.

ox_exchange_mathcap
:   If 'on' Asir forces the exchange of mathcaps at the communication startup. The default value is 'on'. See Chapter 7 [Distributed computation], page 102.

debug_window
:   When 1, a debug window pops up. (The default value is 0.)

no_debug_on_error
> When 1, asir or ox_asir do not get into the debug mode.

fake_ox_reset
> When 1, the function ox_reset sends ox_shutdown insread of ox_reset to
> servers which do not implement ox_reset protocol. It is checked by the op-
> tional argument (the fourth argument) containing the string "no_ox_reset"
> of mathcap if ox_reset is implemented or not. When you use ox_pari, it is
> recommended to set this flag 1. (The default value is 0.) See Chapter 7
> [Distributed computation], page 102.

parallel_gc
> When the value is 0, GC works in the main thread. When the value is n,
> GC works in n threads. (Default value depends on a machine.)

plot_by_bigfloat
> If its value N is not equal to zero, then plotting is done by bigfloat. (The
> double numbers are used as the default.)

evalef   When it is set to 0, the arguments of eval are registered to the list of inde-
> terminants when arguments are no longer to be simplified. For example,
> when this flag is 0, $\sin(0.1)$ is registered to the list of indeterminants when
> $\mathrm{eval}(\sin(0.1))$ is evaluated. When it is set to 1, arguments are numerically
> evaluated. It is recommended to turn on this option, when you repeat to
> evaluate elementary functions numerically. (The default value is 0.) See
> Section 6.3.7 [ord], page 49,

```
[1917] ctrl();
[[oxpari_start,0],[oxpari_id,-1],... snip ....,[plot_by_bigfloat,0],[debug_plot,0]]█
[1918] map(print,ctrl(|desc=1));
[oxpari_start,0,oxpari starting function.]
[oxpari_id,-1,Id of ox_pari.]
... snip ...
[plot_by_bigfloat,0,If set to 1, computation is done by using MPFR bigfloat functions
[debug_plot,0,If set to 1, ox_plot is executed with the message window.]
```

References
> Section 6.14.6 [cputime tstart tstop], page 96, Section 6.11.4 [output],
> page 84, Section 6.1.14 [pari], page 41, Section 6.1.15 [setbprec setprec],
> page 42, Section 6.1.13 [eval deval], page 40.

ChangeLog

- See Section A.5.1 [ChangeLog], page 197,
- plot_by_bigfloat switch is added on 2017.09.04. The following files are changed.
- 1.54 ctrl.c, 1.14 bf.c, 1.75 tcpf.c, 1.13 calc.c, 1.36 if.c, 1.25 ifplot.h, 1.33 plotf.c

6.14.2 debug

debug   :: Forces to enter into debugging mode.

Function debug is a function with no argument. It can be called without '()'.

- In the debug-mode, you are prompted by (debug) and the debugger is ready for com-
  mands. Typing in quit (Note! without a semicolon.) brings you to exit the debug-
  mode.
- See Chapter 5 [Debugger], page 31, for details.

```
[1] debug;
(debug) quit
0
[2]
```

6.14.3 error

`error(message)`

          :: Forces **Asir** to cause an error and enter debugging mode.

message   string

- When **Asir** encounters a serious error such that it finds difficult to continue execution,
  it, in general, tries to enter debugging mode before it returns to top level. The command
  `error()` forces a similar behavior in a user program.
- The argument is a string which will be displayed when `error()` will be executed.
- You can enter the debug-mode when your program encounters an illegal value for a
  program variable, if you have written the program so as to call `error()` upon finding
  such an error in your program text.

```
% cat mod3
def mod3(A) {
        if ( type(A) >= 2 )
                error("invalid argument");
        else
                return A % 3;
}
end$
% asir
[0] load("mod3");
1
[3] mod3(5);
2
[4] mod3(x);
invalid argument
stopped in mod3 at line 3 in file "./mod3"
3                       error("invalid argument");
(debug) print A
A = x
(debug) quit
return to toplevel
[4]
```

References

        Section 6.14.2 [debug], page 94.

6.14.4 `help`

`help(["`*function*`"])`

:: Displays the description of function *function*.

return   0

function   function name

- If invoked without argument, it displays rough usage of **Asir**.
- If a function name is given and if there exists a file with the same name in the directory '`help`' under standard library directory, the file is displayed by a command set to the environmental variable `PAGER` or else command '`more`'.
- If the `LANG` environment variable is set and its value begins with `"japan"` or `"ja_JP"`, then the file in '`help-ja`' is displayed. If its value does not begin with `"japan"` or `"ja_JP"`, then the file in '`help-en`' is displayed.
- On Windows HTML-style help is available from the menu.

6.14.5 `time`

`time()`   :: Returns a four element list consisting of total CPU time, GC time(current version displays 0), the elapsed time and also total memory quantities requested from the start of current **Asir** session.

return   list

- These are commands regarding CPU time and GC time (the current version adds the GC time to the CPU time).
- The GC time is the time regarded to spent by the garbage collector, and the CPU time is the time defined by subtracting the GC time from the total time consumed by command **Asir**. Their unit is 'second.' (The current version adds the GC time to the CPU time, then the GC time shows 0.)
- Command `time()` returns total CPU time and GC time (0 is displayed) measured from the start of current **Asir** session. It also returns the elapsed time. Time unit is 'second.' Moreover, it returns total memory quantities in words (usually 4 bytes) which are requested to the memory manager from the beginning of the current session. The return value is a list and the format is `[CPU time, GC time, Memory, Elapsed time]`.
- You can find the CPU time and GC time for some computation by taking the difference of the figure reported by `time()` at the beginning and the ending of the computation.
- Since arbitrary precision integers are NOT used for counting the total amount of memory request, the number will eventually happen to become meaningless due to integer overflow.
- When `cputime` switch is active by `ctrl()` or by `cputime()`, the execution time will be displayed after every evaluation of top level statement. In a program, however, in order to know the execution time for a sequence of computations, you have to use `time()` command, for an example.
- On UNIX, if `getrusage()` is available, `time()` reports reliable figures. On Windows NT it also gives reliable CPU time. However, on Windows 95/98, the reported time is nothing but the elapsed time of the real world. Therefore, the time elapsed in the

debug-mode and the time of waiting for a reply to interruption prompting are added
to the elapsed time.

```
[72] T0=time();
[2.390885,0.484358,46560,9.157768]
[73] G=hgr(katsura(4),[u4,u3,u2,u1,u0],2)$
[74] T1=time();
[8.968048,7.705907,1514833,63.359717]
[75] ["CPU",T1[0]-T0[0],"GC",T1[1]-T0[1]];
[CPU,6.577163,GC,7.221549]
```

References
        Section 6.14.6 [cputime tstart tstop], page 96, Section 6.14.8 [currenttime],
        page 97.

6.14.6 cputime, tstart, tstop

cputime(*onoff*)
        :: Stop displaying cputime if its argument is 0, otherwise start displaying
        cputime after every top level evaluation of Asir command.

tstart()   :: Resets and starts timer for CPU time and GC time.

tstop()   :: Stops timer and then displays CPU time GC time elapsed from the last time
        when timer was started.

return   0

onoff   flag (arbitrary)

- Command cputime() with NON-ZERO argument enables **Asir** to display CPU time
  and GC time after every evaluation of top level **Asir** command. The command with
  argument 0 disables displaying them.

- Command tstart() starts measuring CPU time and GC time without arguments. The
  parentheses '()' may be omitted.

- Command tstop() stops measuring CPU time and GC time and displays them without
  arguments. The parentheses '()' may be omitted.

- Command cputime(*onoff*) has same meaning as ctrl("cputime",*onoff*).

- Nested use of tstart() and tstop() is not expected. If such an effect is desired, use
  time().

- On and off states by cputime() have effects only to displaying mode. Time for eval-
  uation of every top level statement is always measured. Therefore, even after a com-
  putation has already started, you can let **Asir** display the timings, whenever you enter
  the debug-mode and execute cputime(1).

```
[49] tstart$
[50] fctr(x^10-y^10);
[[1,1],[x+y,1],[x^4-y*x^3+y^2*x^2-y^3*x+y^4,1],[x-y,1],
[x^4+y*x^3+y^2*x^2+y^3*x+y^4,1]]
[51] tstop$
```

```
    80msec + gc : 40msec
```
References
>   Section 6.14.5 [`time`], page 95, Section 6.14.8 [`currenttime`], page 97, Section 6.14.1 [`ctrl`], page 91.

6.14.7 `timer`

`timer(`*interval*`,`*expr*`,`*val*`)`
>   :: Compute an expression under the interval timer.

return   result

interval   interval (second)

expr   expression to be computed

val   a value to be returned when the timer is expired

- `timer()` computes an expression under the interval timer. If the computation finishes within the specified interval, it returns the result of the computation. Otherwise it returns the third argument.
- The third argument should be distinguishable from the result on success.
  ```
  [0] load("cyclic");
  1
  [10] timer(10,dp_gr_main(cyclic(7),[c0,c1,c2,c3,c4,c5,c6],1,1,0),0);
  interval timer expired (VTALRM)
  0
  [11]
  ```

6.14.8 `currenttime`

`currenttime()`
>   :: Get current time.

return   UNIX time.

- See also time(3) in UNIX manuals.
  ```
  [0] currenttime();
  1071639228
  [1]
  ```

6.14.9 `sleep`

`sleep(`*interval*`)`
>   :: Suspend computation for an interval.

return   1

interval   interval (micro second)

- See also usleep(3) in UNIX manuals.
  ```
  [0] sleep(1000);
  1
  [1]
  ```

6.14.10 `heap`

`heap()`   :: Heap area size currently in use.

return   non-negative integer

- Command `heap()` returns an integer which is the byte size of current **Asir** heap area.

  Heap is a memory area where various data for expressions and user programs of **Asir** and is managed by the garbage collector. While **Asir** is running, size of the heap is monotonously non-decreasing against the time elapsed. If it happens to exceed the real memory size, most (real world) time is consumed for swapping between real memory and disk memory.

- For a platform with little real memory, it is recommended to set up **Asir** configuration tuned for GC functions by `-adj` option at the activation of **Asir**. (See Section 2.4 [Command line options], page 6.)

```
% asir -adj 16
[0] load("fctrdata")$
0
[97] cputime(1)$
0msec
[98] heap();
524288
0msec
[99] fctr(Wang[8])$
3.190sec + gc : 3.420sec
[100] heap();
1118208
0msec
[101] quit;
% asir
[0] load("fctrdata")$
0
[97] cputime(1)$
0msec
[98] heap();
827392
0msec
[99] fctr(Wang[8])$
3.000sec + gc : 1.180sec
[100] heap();
1626112
0msec
[101] quit;
```

References
        Section 2.4 [Command line options], page 6.

6.14.11 `version`

`version()`

:: Version identification number of **Asir**.

return   integer

- Command `version()` returns the version identification number, an integer of **Asir** in use.

```
[0] version();
991214
```

6.14.12 `shell`

`shell(command)`

:: Execute shell commands described by a string *command*.

return   integer

command   string

Execute shell commands described by a string *command* by a C function `system()`. This returns the exit status of shell as its return value.

```
[0] shell("ls");
alg             da              katsura         ralg            suit
algt            defs.h          kimura          ratint          test
alpi            edet            kimura3         robot           texput.log
asir.o          fee             mfee            sasa            wang
asir_symtab     gr              mksym           shira           wang_data
base            gr.h            mp              snf1            wt
bgk             help            msubst          solve
chou            hom             p               sp
const           ifplot          proot           strum
cyclic          is              r               sugar
0
[1]
```

6.14.13 `map`

`map(function,arg0,arg1,...)`

:: Applies a function to each member of a list or an array.

return   an object of the same type as *arg0*.

function   the name of a function

arg0   list, vector or matrix

arg1 ...   arbitrary (the rest of arguments)

- Returns an object of the same type as *arg0*. Each member of the returned object is the return value of a function call where the first argument is the member of *arg0* corresponding to the member in the returned object and the rest of the argument are *arg1*, ...

- *function* is a function name itself without '"'.
- A program variable cannot be used as *function*.
- If *arg0* is neither list nor array this function simply returns the value of *function*(*arg0,arg1,...*).
  ```
  [82] def afo(X) { return X^3; }
  [83] map(afo,[1,2,3]);
  [1,8,27]
  ```

6.14.14 `flist`

`flist([mname])`
::: Returns the list of function names currently defined.

`mname`   a module name

return   list of character strings
- Returns the list of names of built-in functions and user defined functions currently defined. The return value is a list of character strings. If a module name is specified, only the names of functions defined in the module are returned.
- The names of built-in functions are followed by those of user defined functions.
  ```
  [77] flist();
  [defpoly,newalg,mainalg,algtorat,rattoalg,getalg,alg,algv,...]
  ```

6.14.15 `delete_history`

`delete_history([index])`
::: Deletes the history.

return   0

index   Index of history to be deleted.
- Deletes all the histories without an argument.
- Deletes the history with index *index* if specified.
- A history is an expression which has been obtained by evaluating an input given for a prompt with an index. It can be taken out by `@index`, which means that the expression survives garbage collections.
- A large history may do harm in the subsequent memory management and deleting the history by `delete_history()`, after saving it in a file by `bsave()`, is often effective.
  ```
  [0] (x+y+z)^100$
  [1] @0;
  ...
  [2] delete_history(0);
  [3] @0;
  0
  ```

6.14.16 `get_rootdir`

`get_rootdir()`
::: Gets the name of **Asir** root directory.

return   string
- On UNIX it returns the value of an environment variable `ASIR_LIBDIR` or '`/usr/local/lib/asir`' if `ASIR_LIBDIR` is not set.
- On Windows the name of **Asir** root directory is returned.
- By using relative path names from the value of this function, one can write programs which contain file operations independent of the install directory.

6.14.17 `getopt`

`getopt([`*key*`])`
:: Returns the value of an option.

return   object
- When a user defined function is called, the number of arguments must be equal to that in the declaration of the function. A function with indefinite number of arguments can be realized by using *options* (see Section 4.2.12 [option], page 27). The value of a specified option is retrieved by `getopt`.
- If `getopt()` is called with no argument, then it returns a list `[[key1,value1], [key2,value2],...]`. In the list, each `key` is an option which was specified when the function executing `getopt` was invoked, and `value` is the value of the option.
- If an option *key* is specified upon a function call, `getopt` return the value of the option. If such an option is not specified, the it returns an object of VOID type whose object identifier is -1. By examining the type of the returned value with `type()`, one knows whether the option is set or not.
- Options are specified as follows:

      `xxx(A,B,C,D|x=X,y=Y,z=Z)`

  That is, the options are specified by a sequence of *key=value* separated by '`,`', after '`|`'.

`References`
:: Section 4.2.12 [option], page 27, Section 6.8.1 [type], page 76.

6.14.18 `getenv`

`getenv(`*name*`)`
:: Returns the value of an environment variable.

return
name   string
- Returns the value of an environment variable *name*.

      ```
      [0] getenv("HOME");
      /home/pcrf/noro
      ```

7 Distributed computation

7.1 OpenXM

On **Asir** distributed computations are done under **OpenXM** (Open message eXchange protocol for Mathematics), which is a protocol for exchanging mainly mathematical objects between processes. See `http://www.math.sci.kobe-u.ac.jp/OpenXM/` for the details of **OpenXM**. In **OpenXM** a distributed computation is done as follows:

1. A client requests something to a server.

2. The server does works according to the request.

3. The client requests to send data to the server.

4. The server sends the data to the client and the client gets the data.

The server is a stack machine. That is data objects sent by the client are pushed to the stack of the server. If the server gets a command, then the data are popped form the stack and they are used as arguments of a function call.

In **OpenXM**, the result of a computation done in the server is simply pushed to the stack and the data is not written to the communication stream without requests from the client.

OpenXM protocol consists of two components: **CMO** (Common Mathematical Object format) which determines a common format of data representations and **SM** (StackMachine command) which specifies actions on servers. These are wrapped as **OX** expressions to indicate the sort of data when they are sent.

To execute a distributed computation by **OpenXM**, one has to invoke **OpenXM** servers and to establish communications between the client and the servers. `ox_launch()`, `ox_launch_nox()`, `ox_launch_generic()` are prepared for such purposes. Furthermore the following functions are available.

`ox_push_cmo()`

> It requests a server to push an object to the stack of a server.

`ox_pop_cmo()`

> It request a server to pop an object from the stack of a server.

`ox_cmo_rpc()`

> It requests to execute a function on a server. The result is pushed to the stack of the server.

`ox_execute_string()`

> It requests a server to parse and execute a string by the parser and the evaluator of the server. The result is pushed to the stack of the server.

`ox_push_cmd()`

> It requests a server to execute a command.

`ox_get()`

> It gets an object from a data stream.

7.2 Mathcap

A server or a client does not necessarily implement full specifications of **OpenXM**. If a program sends data unknown to its peer, an unrecoverable error may occur. To avoid such a case **OpenXM** provides a scheme not to send data unknown to peers. It is realized by exchanging the list of supported **CMO** and **SM**. The list is called mathcap. Mathcap is also defined as a **CMO** and the elements are 32bit integers or strings. The format of mathcap is as follows.

[[version number, server name],**SM**taglist, [[**OX**tag,**CMO**taglist],[**OX**tag,**CMO**taglist],...]]

[**OX**tag,**CMO**taglist] indicates that available object tags for a category of data specified by **OX**tag. For example 'ox_asir' accepts the local object format used by **Asir** and the mathcap from 'ox_asir' reflects the fact.

If "ox_check" switch of ctrl is set to 1, the check by a mathcap is done before data is sent. If "ox_check" switch of ctrl is set to 0, the check is not done. By default it is set to 1.

7.3 Stackmachine commands

The stackmachine commands are provided to request a server to execute various operations. They are automatically sent by built-in functions of **Asir**, but one often has to send them manually. They are represented by 32bit integers. One can send them by calling ox_push_cmd(). Typical stackmachine commands are as follows. **SM_xxx=yyy** means that **SM_xxx** is a mnemonic and that **yyy** is its value.

SM_popSerializedLocalObject=258

An object not necessarily defined as **CMO** is popped from the stack and is sent to the client. This is available only on 'ox_asir'.

SM_popCMO=262

A **CMO** object is popped from the stack and is sent to the client.

SM_popString=263

An object is popped from the stack and is sent to the client as a readable string.

SM_mathcap=264

The server's mathcap is pushed to the stack.

SM_pops=265

Objects are removed from the stack. The number of object to be removed is specified by the object at the top of the stack.

SM_setName=266

A variable name is popped form the stack. Then an object is popped and it is assigned to the variable. This assignment is done by the local language of the server.

SM_evalName=267

A variable name is popped from the stack. Then the value of the variable is pushed to the stack.

SM_executeStringByLocalParser=268

A string popped from the stack is parsed and evaluated. The result is pushed to the stack.

SM_executeFunction=269

> A function name, the number of arguments and the arguments are popped from the stack. Then the function is executed and the result is pushed to the stack.

SM_beginBlock=270

> It indicates the beginning of a block.

SM_endBlock=271

> It indicates the end of a block.

SM_shutdown=272

> It shuts down communications and terminates servers.

SM_setMathcap=273

> It requests a server to register the data at the top of the stack as the client's mathcap.

SM_getsp=275

> The number of objects in the current stack is pushed to the stack.

SM_dupErrors=276

> The list of all the error objects in the current stack is pushed to the stack.

SM_nop=300

> Nothing is done.

7.4 Debugging

In general, it is difficult to debug distributed computations. '`ox_asir`' provides several functions for debugging.

7.4.1 Error object

When an error has occurred on an **OpenXM** server, an error object is pushed to the stack instead of a result of the computation. The error object consists of the serial number of the **SM** command which caused the error, and an error message.

```
[340] ox_launch();
0
[341] ox_rpc(0,"fctr",1.2*x);
0
[342] ox_pop_cmo(0);
error([8,fctrp : invalid argument])
```

7.4.2 Resetting a server

`ox_reset()` resets a process whose identifier is *number*. After its execution the process is ready for receiving data. This function corresponds to the keyboard interrupt on an usual **Asir** session. It often happens that a request of a client does not correspond correctly to the result from a server. It is caused by remaining data on data streams. `ox_reset` is effective for such cases.

7.4.3 Pop-up command window for debugging

As a server does not have any standard input device such as a keyboard, it is difficult to debug user programs running on the server. 'ox_asir' pops up a small command window to input debug commands when an error has occurred during user a program execution or ox_rpc(*id*,"debug") has been executed. The responses to commands are shown in 'xterm' to display standard outputs from the server. To close the small window, input quit.

7.5 Functions for distributed computation

7.5.1 ox_launch, ox_launch_nox, ox_shutdown

ox_launch([*host*[,*dir*],*command*])
ox_launch_nox([*host*[,*dir*],*command*])
            :: Initialize OpenXM servers.

ox_shutdown(*id*)
            :: Terminates OpenXM servers.

return   integer

host   string or 0

dir command

                string

id   integer

- Function ox_launch() invokes a process to execute *command* on a host *host* and enables **Asir** to communicate with that process. If the number of arguments is 3, 'ox_launch' in *dir* is invoked on *host*. Then 'ox_launch' invokes *command*. If *host* is equal to 0, all the commands are invoked on the same machine as the **Asir** is running. If no arguments are specified, *host*, *dir* and *command* are regarded as 0, the value of get_rootdir() and 'ox_asir' in the same directory respectively.

- If *host* is equal to 0, then *dir* can be omitted. In such a case *dir* is regarded as the value of get_rootdir().

- If *command* begins with '/', it is regarded as an absolute pathname. Otherwise it is regarded as a relative pathname from *dir*.

- On UNIX, ox_launch() invokes 'xterm' to display standard outputs from *command*. If X11 is not available or one wants to invoke servers without 'xterm', use ox_launch_nox(), where the outputs of *command* are redirected to '/dev/null'. If the environment variable DISPLAY is not set, ox_launch() and ox_launch_nox() behave identically.

- The returned value is used as the identifier for communication.

- The peers communicating with **Asir** are not necessarily processes running on the same machine. The communication will be successful even if the byte order is different from those of the peer processes, because the byte order for the communication is determined by a negotiation between a client and a server.

- The following preparations are necessary. Here, Let A be the host on which **Asir** is running, and B the host on which the peer process will run.

1. Register the hostname of the host `A` to the '`~/.rhosts`' of the host `B`. That is, you should be allowed to access the host `B` from `A` without supplying a password.

2. For cases where connection to `X` is also used, let `Xserver` authorize the relevant hosts. Adding the hosts can be done by command `xhost`.

3. If an environment variable `ASIR_RSH` is set, the content of this variable is used as a promgram to invoke remote servers instead of *rsh*. For example,

   ```
   % setenv ASIR_RSH "ssh -f -X -A "
   ```

   implies that remote servers are invoked by '`ssh`' and that X11 forwarding is enabled. See the manual of '`ssh`' for the detail.

4. Some *command*'s consume much stack space. You are recommended to set the stack size to about 16MB large in '`.cshrc`' for safe. To specify the size, put `limit stacksize 16m` for an example.

- When *command* opens a window on `X`, it uses the string specified for *display*; if the specification is omitted, it uses the value set for the environment variable `DISPLAY`.

- `ox_shutdown()` terminates OpenXM servers whose identifier is *id*.

- When **Asir** is terminated successfully, all I/O streams are automatically closed, and all the processes invoked are also terminated. However, some remote processes may not terminated when **Asir** is terminated abnormally. If ever **Asir** is terminated abnormally, you have to kill all the unterminated process invoked by **Asir** on every remote host. Check by `ps` command on the remote hosts to see if such processed are alive.

- '`xterm`' for displaying the outputs from *command* is invoked with '`-name ox_term`' option. Thus, by specifying resources for the resource name '`ox_term`', only the behaviour of the '`xterm`' can be customized.

  ```
  /* iconify on start */
  ox_xterm*iconic:on
  /* activate the scroll bar */
  ox_xterm*scrollBar:on
  /* 1000 lines can be shown by the scrollbar */
  ox_xterm*saveLines:1000
  ```

  ```
  [219] ox_launch();
  0
  [220] ox_rpc(0,"fctr",x^10-y^10);
  0
  [221] ox_pop_local(0);
  [[1,1],[x^4+y*x^3+y^2*x^2+y^3*x+y^4,1],
  [x^4-y*x^3+y^2*x^2-y^3*x+y^4,1],[x-y,1],[x+y,1]]
  [222] ox_shutdown(0);
  0
  ```

References

Section 7.5.5 [`ox_rpc ox_cmo_rpc ox_execute_string`], page 110, Section 7.5.8 [`ox_pop_cmo ox_pop_local`], page 112, Section 7.5.15 [`ifplot conplot plot polarplot plotover`], page 116

7.5.2 `ox_launch_generic`

`ox_launch_generic(host,launch,server,use_unix,use_ssh,use_x,conn_to_serv)`
          :: Initialize OpenXM servers.

return   integer

host   string or 0

launcher server
          string

use_unix use_ssh use_x conn_to_serv
          integer

- `ox_launch_generic()` invokes a control process *launch* and a server process *server* on *host*. The other arguments are switches for protocol family selection, on/off of the X environment, method of process invocation and selection of connection type.

- If *host* is equal to 0, processes are invoked on the same machine as the **Asir** is running. In this case UNIX internal protocol is always used.

- If *use_unix* is equal to 1, UNIX internal protocol is used. If *use_unix* is equal to 0, Internet protocol is used.

- If *use_ssh* is equal to 1,'`ssh`' (Secure Shell) is used to invoke processes. If one does not use '`ssh-agent`', a password (passphrase) is required. If '`sshd`' is not running on the target machine, '`rsh`' is used instead. But it will immediately fail if a password is required.

- If *use_x* is equal to 1, it is assumed that X environment is available. In such a case *server* is invoked under '`xterm`' by using the current `DISPLAY` variable. If `DISPLAY` is not set, it is invoked without X. Note that the processes will hang up if `DISPLAY` is incorrectly set.

- If *conn_to_serv* is equal to 1, **Asir** (client) executes `bind` and `listen`, and the invoked processes execute `connect`. If *conn_to_serv* is equal to 0, **Asir** (client) the invoked processes execute `bind` and `listen`, and the client executes `connect`.

```
[342] LIB=get_rootdir();
/export/home/noro/ca/Kobe/build/OpenXM/lib/asir
[343] ox_launch_generic(0,LIB+"/ox_launch",LIB+"/ox_asir",0,0,0,0);
1
[344] ox_launch_generic(0,LIB+"/ox_launch",LIB+"/ox_asir",1,0,0,0);
2
[345] ox_launch_generic(0,LIB+"/ox_launch",LIB+"/ox_asir",1,1,0,0);
3
[346] ox_launch_generic(0,LIB+"/ox_launch",LIB+"/ox_asir",1,1,1,0);
4
[347] ox_launch_generic(0,LIB+"/ox_launch",LIB+"/ox_asir",1,1,1,1);
5
[348] ox_launch_generic(0,LIB+"/ox_launch",LIB+"/ox_asir",1,1,0,1);
6
```

References

Section 7.5.1 [ox_launch ox_launch_nox ox_shutdown], page 105,
Section 7.5.2 [ox_launch_generic], page 107

7.5.3 generate_port, try_bind_listen, try_connect, try_accept, register_server

generate_port([*use_unix*])

          :: Generates a port number.

try_bind_listen(*port*)

          :: Binds and listens on a port.

try_connect(*host*,*port*)

          :: Connects to a port.

try_accept(*socket*,*port*)

          :: Accepts a connection request.

register_server(*control_socket*,*control_port*,*server_socket*,*server_port*)

          :: Registers the sockets for which connections are established.

return   integer or string for generate_port(), integer for the others

use_unix   0 or 1

host   string

port control_port server_port

          integer or string

socket control_socket server_socket

          integer

- These functions are primitives to establish communications between a client and servers.
- generate_port() generates a port name for communication. If the argument is not specified or equal to 0, a port number for Internet domain socket is generated randomly. Otherwise a file name for UNIX domain (host-internal protocol) is generated. Note that it is not assured that the generated port is not in use.
- try_bind_listen() creates a socket according to the protocol family indicated by the given port and executes bind and listen. It returns a socket identifier if it is successful. -1 indicates an error.
- try_connect() tries to connect to a port *port* on a host *host*. It returns a socket identifier if it is successful. -1 indicates an error.
- try_accept() accepts a connection request to a socket *socket*. It returns a new socket identifier if it is successful. -1 indicates an error. In any case *socket* is automatically closed. *port* is specified to distinguish the protocol family of *socket*.
- register_server() registers a pair of a control socket and a server socket. A process identifier indicating the pair is returned. The process identifier is used as an argument of ox functions such as ox_push_cmo().
- Servers are invoked by using shell(), or manually.

```
[340] CPort=generate_port();
39896
[341] SPort=generate_port();
37222
[342] CSocket=try_bind_listen(CPort);
3
[343] SSocket=try_bind_listen(SPort);
5


/*
ox_launch is invoked here :
%  ox_launch "127.1" 0 39716 37043 ox_asir "shio:0"
*/

[344] CSocket=try_accept(CSocket,CPort);
6
[345] SSocket=try_accept(SSocket,SPort);
3
[346] register_server(CSocket,CPort,SSocket,SPort);
0
```

References
    Section   7.5.1   [ox_launch ox_launch_nox ox_shutdown],   page   105,
    Section 7.5.2 [ox_launch_generic], page 107, Section 6.14.12 [shell], page 99,
    Section 7.5.7 [ox_push_cmo ox_push_local], page 111

7.5.4 'ox_asir'

'ox_asir' provides almost all the functionalities of **Asir** as an **OpenXM** server. 'ox_asir'
is invoked by ox_launch or ox_launch_nox. If X environment is not available or is not
necessary, one can use ox_launch_nox.

```
[5] ox_launch();
0

[5] ox_launch_nox("127.0.0.1","/usr/local/lib/asir",
"/usr/local/lib/asir/ox_asir");
0

[7] RemoteLibDir = "/usr/local/lib/asir/"$
[8] Machines = ["sumire","rokkaku","genkotsu","shinpuku"];
[sumire,rokkaku,genkotsu,shinpuku]
[9] Servers = map(ox_launch,Machines,RemoteLibDir,
RemoteLibDir+"ox_asir");
[0,1,2,3]
```

References
    Section 7.5.1 [ox_launch ox_launch_nox ox_shutdown], page 105

7.5.5 `ox_rpc, ox_cmo_rpc, ox_execute_string`

`ox_rpc(`*number*`,"`*func*`",`*arg0*`,...)`
`ox_cmo_rpc(`*number*`,"`*func*`",`*arg0*`,...)`
`ox_execute_string(`*number*`,"`*command*`",...)`
        :: Calls a function on an OpenXM server

return   0

number   integer (process identifier)

func   function name

command   string

arg0 ...   arbitrary (arguments)

- Calls a function on an **OpenXM** server whose identifier is *number*.
- It returns 0 immediately. It does not wait the termination of the function call.
- `ox_rpc()` can be used when the server is 'ox_asir'. Otherwise `ox_cmo_rpc()` should be used.
- The result of the function call is put on the stack of the server. It can be received by `ox_pop_local()` or `ox_pop_cmo()`.
- If the server is not 'ox_asir', only data defined in **OpenXM** can be sent.
- `ox_execute_string` requests the server to parse and execute *command* by the parser and the evaluater of the server. The result is pushed to the stack.

```
[234] ox_cmo_rpc(0,"dp_ht",dp_ptod((x+y)^10,[x,y]));
0
[235] ox_pop_cmo(0);
(1)*<<10,0>>
[236] ox_execute_string(0,"12345 % 678;");
0
[237] ox_pop_cmo(0);
141
```

`References`
        Section 7.5.8 [ox_pop_cmo ox_pop_local], page 112

7.5.6 `ox_reset,ox_intr,register_handler`

`ox_reset(`*number*`)`
        :: Resets an OpenXM server

`ox_intr(`*number*`)`
        :: Sends `SIGINT` to an OpenXM server

`register_handler(`*func*`)`
        :: Registers a function callable on a keyboard interrupt.

return   1

number   integer(process identifier)

func   functor or 0

- `ox_reset()` resets a process whose identifier is *number*. After its execution the process is ready for receiving data.
- After executing `ox_reset()`, sending/receiving buffers and stream buffers are assured to be empty.
- Even if a process is running, the execution is safely stopped.
- `ox_reset()` may be used prior to a distributed computation. It can be also used to interrupt a distributed computation.
- `ox_intr()` sends `SIGINT` to a process whose identifier is *number*. The action of a server against `SIGINT` is not specified in **OpenXM**. '`ox_asir`' immediately enters the debug mode and pops up an window to input debug commands on X window system.
- `register_handler()` registers a function *func*(). If `u` is specified on a keyboard interrupt, *func*() is executed before returning the toplevel. If `ox_reset()` calls are included in *func*(), one can automatically reset **OpenXM** servers on a keyboard interrupt.
- If *func* is equal to 0, the setting is reset.

  ```
  [10] ox_launch();
  0
  [11] ox_rpc(0,"fctr",x^100-y^100);
  0
  [12] ox_reset(0); /* usr1 : return to toplevel by SIGUSR1 */
  1                 /* is displayed on the xterm.          */
  [340] Procs=[ox_launch(),ox_launch()];
  [0,1]
  [341] def reset() { extern Procs; map(ox_reset,Procs);}
  [342] map(ox_rpc,Procs,"fctr",x^100-y^100);
  [0,0]
  [343] register_handler(reset);
  1
  [344] interrupt ?(q/t/c/d/u/w/?) u
  Abort this computation? (y or n) y
  Calling the registered exception handler...done.
  return to toplevel
  ```

References

7.5.7 ox_push_cmo, ox_push_local

`ox_push_cmo(`*number*`,`*obj*`)`
`ox_push_local(`*number*`,`*obj*`)`
            :: Sends *obj* to a process whose identifier is *number*.

return   0

number   integer(process identifier)

obj   object

- Sends *obj* to a process whose identifier is *number*.
- `ox_push_cmo` is used to send data to an **Open_XM** other than '`ox_asir`' and '`ox_plot`'.

- `ox_push_local` is used to send data to 'ox_asir' and 'ox_plot'.
- The call immediately returns unless the stream buffer is full.

References

7.5.8 ox_pop_cmo, ox_pop_local

`ox_pop_local(`*number*`)`

       :: Receives data from a process whose identifier is *number*.

return   received data

number   integer(process identifier)

- Receives data from a process whose identifier is *number*.
- `ox_pop_cmo` can be used to receive data form an **OpenXM** server other than 'ox_asir' and 'ox_plot'.
- `ox_pop_local` can be used to receive data from 'ox_asir', 'ox_plot'.
- If no data is available, these functions block. To avoid it, send `SM_popCMO` (262) or `SM_popSerializedLocalObject` (258). Then check the process status by `ox_select`. Finally call `ox_get` for a ready process.

```
[341] ox_cmo_rpc(0,"fctr",x^2-1);
0
[342] ox_pop_cmo(0);
[[1,1],[x-1,1],[x+1,1]]
[343] ox_cmo_rpc(0,"newvect",3);
0
[344] ox_pop_cmo(0);
error([41,cannot convert to CMO object])
[345] ox_pop_local(0);
[ 0 0 0 ]
```

References

7.5.9 ox_push_cmd, ox_sync

`ox_push_cmd(`*number*`,`*command*`)`

       :: Sends a command *command* to a process whose identifier is *number*.

`ox_sync(`*number*`)`

       :: Sends **OX_SYNC_BALL** to a process whose identifier is *number*.

return   0

number   integer(process identifier)

command   integer(command identifier)

- Sends a command or **OX_SYNC_BALL** to a process whose identifier is *number*.

- Data in **OpenXM** are categorized into three types: **OX_DATA**, **OX_COMMAND**, **OX_SYNC_BALL**. Usually **OX_COMMAND** and **OX_SYNC_BALL** are sent implicitly with high level operations, but these functions are prepared to send these data explicitly.

- **OX_SYNC_BALL** is used on the reseting operation by `ox_reset`. Usually **OX_SYNC_BALL** will be ignored by the peer.

```
[3] ox_rpc(0,"fctr",x^100-y^100);
0
[4] ox_push_cmd(0,258);
0
[5] ox_select([0]);
[0]
[6] ox_get(0);
[[1,1],[x^2+y^2,1],[x^4-y*x^3+y^2*x^2-y^3*x+y^4,1],...]
```

References

> Section 7.5.5 [`ox_rpc ox_cmo_rpc ox_execute_string`], page 110, Section 7.5.6 [`ox_reset ox_intr register_handler`], page 110

7.5.10 `ox_get`

`ox_get(`*number*`)`

> :: Receives data form a process whose identifier is *number*.

return

number   integer(process identifier)

- Receives data form a process whose identifier is *number*.

- One may use this function with `ox_push_cmd`.

- `ox_pop_cmo` and `ox_pop_local` is realized as combinations of `ox_push_cmd` and `ox_get`.

```
[11] ox_push_cmo(0,123);
0
[12] ox_push_cmd(0,262); /* 262=OX_popCMO */
0
[13] ox_get(0);
123
```

References

> Section 7.5.8 [`ox_pop_cmo ox_pop_local`], page 112, Section 7.5.9 [`ox_push_cmd ox_sync`], page 112

7.5.11 `ox_pops`

`ox_pops(`*number*`[,`*nitem*`])`

> :: Removes data form the stack of a process whose identifier is *number*.

return   0

number   integer(process identifier)

nitem   non-negative integer

- Removes data form the stack of a process whose identifier is *number*. If *nitem* is specified, *nitem* items are removed. If *nitem* is not specified, 1 item is removed.

```
[69] for(I=1;I<=10;I++)ox_push_cmo(0,I);
[70] ox_pops(0,4);
0
[71] ox_pop_cmo(0);
6
```

References

Section 7.5.8 [ox_pop_cmo ox_pop_local], page 112

7.5.12 ox_select

ox_select(*nlist*[,*timeout*])

        :: Returns the list of process identifiers on which data is available.

return   list

nlist   list of integers (process identifier)

timeout   number

- Returns the list of process identifiers on which data is available.
- If all the processes in *nlist* are running, it blocks until one of the processes returns data. If *timeout* is specified, it waits for only *timeout* seconds.
- By sending SM_popCMO or SM_popSerializedLocalObject with ox_push_cmd() in advance and by examining the process status with ox_select(), one can avoid a hanging up caused by ox_pop_local() or ox_pop_cmo(). In such a case, data can be received by ox_get().

```
ox_launch();
0
[220] ox_launch();
1
[221] ox_launch();
2
[222] ox_rpc(2,"fctr",x^500-y^500);
0
[223] ox_rpc(1,"fctr",x^100-y^100);
0
[224] ox_rpc(0,"fctr",x^10-y^10);
0
[225] P=[0,1,2];
[0,1,2]
[226] map(ox_push_cmd,P,258);
[0,0,0]
[227] ox_select(P);
[0]
```

```
[228] ox_get(0);
[[1,1],[x^4+y*x^3+y^2*x^2+y^3*x+y^4,1],
[x^4-y*x^3+y^2*x^2-y^3*x+y^4,1],[x-y,1],[x+y,1]]
```

References
>   Section 7.5.8 [ox_pop_cmo ox_pop_local], page 112, Section 7.5.9 [ox_push_
>   cmd ox_sync], page 112, Section 7.5.10 [ox_get], page 113

7.5.13 ox_flush

ox_flush(`id`)
>   :: Flushes the sending buffer.

return   1

id   process identifier

- By default the batch mode is off and the sending buffer is flushed at every sending operation of data and command.
- The batch mode is set by "ox_batch" switch of "ctrl".
- If one wants to send many pieces of small data, ctrl("ox_batch",1) may decrease the overhead of flush operations. Of course, one has to call ox_flush(`id`) at the end of the sending operations.
- Functions such as ox_pop_cmo and ox_pop_local enter a waiting mode immediately after sending a command. These functions always flush the sending buffer.
  ```
  [340] ox_launch_nox();
  0
  [341] cputime(1);
  0
  7e-05sec + gc : 4.8e-05sec(0.000119sec)
  [342] for(I=0;I<10000;I++)ox_push_cmo(0,I);
  0.232sec + gc : 0.006821sec(0.6878sec)
  [343] ctrl("ox_batch",1);
  1
  4.5e-05sec(3.302e-05sec)
  [344] for(I=0;I<10000;I++)ox_push_cmo(0,I); ox_flush(0);
  0.08063sec + gc : 0.06388sec(0.4408sec)
  [345] 1
  9.6e-05sec(0.01317sec)
  ```

References
>   Section 7.5.8 [ox_pop_cmo ox_pop_local], page 112, Section 6.14.1 [ctrl],
>   page 91

7.5.14 ox_get_serverinfo

ox_get_serverinfo([`id`])
>   :: Gets server's mathcap and proess id.

return   list

id   process identifier

- If *id* is specified, the mathcap of the process whose identifier is *id* is returned.
- If *id* is not specified, the list of [*id,Mathcap*] is returned, where *id* is the identifier of a currently active process, and *Mathcap* is the mathcap of the process. identifier *id* is returned.

```
[343] ox_get_serverinfo(0);
[[199909080,Ox_system=ox_sm1.plain,Version=2.991118,HOSTTYPE=FreeBSD],
[262,263,264,265,266,268,269,272,273,275,276],
[[514],[2130706434,1,2,4,5,17,19,20,22,23,24,25,26,30,31,60,61,27,
33,40,16,34]]]
[344] ox_get_serverinfo();
[[0,[[199909080,Ox_system=ox_sm1.plain,Version=2.991118,
HOSTTYPE=FreeBSD],
[262,263,264,265,266,268,269,272,273,275,276],
[[514],[2130706434,1,2,4,5,17,19,20,22,23,24,25,26,30,31,60,61,27,33,
40,16,34]]]],
[1,[[199901160,ox_asir],
[276,275,258,262,263,266,267,268,274,269,272,265,264,273,300,270,271],
[[514,2144202544],
[1,2,3,4,5,2130706433,2130706434,17,19,20,21,22,24,25,26,31,27,33,60],
[0,1]]]]]
```

```
References
```
Section 7.2 [`Mathcap`], page 103.

7.5.15 `ifplot, conplot, plot, polarplot, plotover`

`ifplot(`*func* `[,`*geometry*`]` `[,`*xrange*`]` `[,`*yrange*`]` `[,`*id*`]` `[,`*name*`])`
:: Displays real zeros of a bi-variate function.

`conplot(`*func* `[,`*geometry*`]` `[,`*xrange*`]` `[,`*yrange*`]` `[,`*zrange*`]` `[,`*id*`]` `[,`*name*`])`
:: Displays real contour lines of a bi-variate function.

`plot(`*func* `[,`*geometry*`]` `[,`*xrange*`]` `[,`*id*`]` `[,`*name*`])`
:: Displays the graph of a univariate function.

`polarplot(`*func* `[,`*geometry*`]` `[,`*thetarange*`]` `[,`*id*`]` `[,`*name*`])`
:: Displays the graph of a curve given in polar form.

`plotover(`*func*`,`*id*`,`*number*`)`
Plots on the existing window real zeros of a bivariate function.

return   integer

func   polynomial

geometry xrange yrange zrange
list

id number integer

name   string

- Function `ifplot()` draws a graph of real zeros of a bi-variate function. Function `conplot()` plots the contour lines for a same argument. Function `plot()` draws the

graph of a uninivariate function. Function `polarplot()` draws the graph of a curve given in polar form $r=f(theta)$.

- The plotting functions are realized by an OpenXM server. On UNIX it is '`ox_plot`' in **Asir** root directory. On Windows '`engine`' acts as '`ox_plot`'. Of course, it must be activated by `ox_launch()` `ox_launch_nox()`. If the identifier of an active '`ox_plot`' is specified as *id*, the server is used for drawing pictures. If *id* is not specified, an available '`ox_plot`' server is used if it exists. If no '`ox_plot`' server is available, then `ox_launch_nox()` is automatically executed to invoke '`ox_plot`'.

- Argument *func* is indispensable. Other arguments are optional. The format of optional arguments and their default values (parenthesized) are listed below.

    geometry   Window size is specified by [*x,y*] in unit 'dot.' [300,300] for UNIX version;

    xrange yrange

            Value ranges of the variables are specified by [*v,vmin,vmax*]. ([*v*,-2,2] for each variable.) If this specification is omitted, the indeterminate having the higher order in *func* is taken for '`x`' and the one with lower order is taken for '`y`'. To change this selection, specify explicitly by *xrange* and/or *yrange*. For an uni-variate function, the specification is mandatory.

    zrange   This specification applies only to `conplot()`. The format is [*v,vmin,vmax* [,*step*]]. If *step* is specified, the height difference of contours is set to (*vmax-vmin*)/*step*. ([*z*,-2,2,16].)

    id   This specifies the number of the remote process by which you wish to draw a graph. (The number for the newest active process.)

    name   The name of the window. (`Plot`.) The created window is titled *name:n/m* which means the *m*-th window of the process with process number *n*. These numbers are used for `plotover()`.

- The maximum number of the windows that can be created on a process is 128.

- Function `plotover()` superposes reals zeros of its argument bi-variate function onto the specified window.

- Enlarged plot can be obtained for rectangular area which is specified, on an already existing window with a graph, by dragging cursor with the left button of mouse from the upper-left corner to lower-right corner and then releasing it. Then, a new window is created whose shape is similar to the specified area and whose size is determined so that the largest side of the new window has the same size of the largest side of the original window. If you wish to cancel the action, drag the cursor to any point above or left of the starting point.

    This facility is effective when `precise` button switch is inactive. If `precise` is selected and active, the area specified by the cursor dragging will be rewritten on the same window. This will be explained later.

- A click of the right button will display the current coordinates of the cursor at the bottom area of the window.

- Place the cursor at any point in the right marker area on a window created by `conplot()`, and drag the cursor with the middle mutton. Then you will find the contour lines changing their colors depending on the movement of the cursor and the corresponding height level displayed on the upper right corner of the window.

- Several operations are available on the window: by button operations for UNIX version, and pull-down menus for Windows version.

  quit   Destroys (kills) the window. While computing, quit the current computa-
            tion. If one wants to interrupt the computation, use `ox_reset()`.

  wide (toggle)
            Will display, on the same window, a new area enlarged by 10 times as
            large as the current area for both width-direction and height-direction.
            The current area will be indicated by a rectangle placed at the center.
            Area specification by dragging the cursor will create a new window with a
            plot of the graph in the specified area.

  precise (toggle)
            When selected and active, `ox_plot` redraws the specified area more pre-
            cisely by integer arithmetic. This mode uses bisection method based on
            Sturm sequence computation to locate real zeros precisely. More precise
            plotting can be expected by this technique than by the default plotting
            technique, at the expense of significant increase of computing time. As
            you see by above explanation, this function is only effective to polynomials
            with rational coefficients. (Check how they differ for (x^2+y^2-1)^2.)

  formula   Displays the expression for the graph.

  noaxis (toggle)
            Erase the coordinates.

- Program '`ox_plot`' may consume much stack space depending on which machine it is
  running. You are recommended to set the stack size to about 16MB as large in '`.cshrc`'
  for safe. To specify the size, put `limit stacksize 16m` for an example.

- You can customize various resources of a window on `X`, e.g., coloring, shape of buttons
  etc. The default setting of resources is shown below. For `plot*form*shapeStyle` you
  can select among `rectangle`, `oval`, `ellipse`, and `roundedRectangle`.

```
plot*background:white
plot*form*shapeStyle:rectangle
plot*form*background:white
plot*form*quit*background:white
plot*form*wide*background:white
plot*form*precise*background:white
plot*form*formula*background:white
plot*form*noaxis*background:white
plot*form*xcoord*background:white
plot*form*ycoord*background:white
plot*form*level*background:white
plot*form*xdone*background:white
plot*form*ydone*background:white
```

References
            Section   7.5.1   [ox_launch ox_launch_nox ox_shutdown],   page   105,
            Section 7.5.6 [ox_reset ox_intr register_handler], page 110

7.5.16 `open_canvas`, `clear_canvas`, `draw_obj`, `draw_string`

`open_canvas(`*id*`[,`*geometry*`])`
:: Opens a canvas, which is a window for drawing objects.

`clear_canvas(`*id*`,`*index*`)`
:: Clears a canvas.

`draw_obj(`*id*`,`*index*`,`*pointorsegment* `[,`*color*`])`
:: Draws a point or a line segment on a canvas.

`draw_string(`*id*`,`*index*`,[`*x*`,`*y*`],`*string* `[,`*color*`])`
:: Draws a character string on a canvas.

return   0

id index color x y
             integer

pointorsegment
             list

string   character string

- These functions are supplied by the OpenXM server 'ox_plot' ('engine' on Windows).
- `open_canvas` opens a canvas, which is a window for drawing objecgts. One can specifies the size of a canvas in pixel by supplying *geometry* option [*x*,*y*]. The default size is [300,300]. This function pushes an integer value onto the stack of the OpenXM server. The value is used to distinguish the opened canvas and one has to pop and maintain the value by `ox_pop_cmo` for subsequent calls of `draw_obj`.
- `clear_canvas` clears a canvas specified by a server id *id* and a canvas id *index*.
- `draw_obj` draws a point or a line segment on a canvas specified by a server id *id* and a canvas id *index*. If *pointorsegment* is [*x*,*y*], it is regarded as a point. If *pointorsegment* is [*x*,*y*,*u*,*v*], it is regarded as a line segment which connects [*x*,*y*] and [*u*,*v*]. If *color* is specified, *color/65536* mod 256, *color/256* mod 256, *color* mod 256 are regarded as the values of Red, Green, Blue (Max. 255) respectively.
- `draw_string` draws a character string *string* on a canvas specified by a server id *id* and a canvas id *index*. The position of the string is specified by [*x*,*y*].

```
[182] Id=ox_launch_nox(0,"ox_plot");
0
[183] open_canvas(Id);
0
[184] Ind=ox_pop_cmo(Id);
0
[185] draw_obj(Id,Ind,[100,100]);
0
[186] draw_obj(Id,Ind,[200,200],0xffff);
0
[187] draw_obj(Id,Ind,[10,10,50,50],0xff00ff);
0
[187] draw_string(Id,Ind,[100,50],"hello",0xffff00);
```

```
    0
    [189] clear_canvas(Id,Ind);
    0
```

References

Section   7.5.1   [ox_launch ox_launch_nox ox_shutdown],   page   105,
Section 7.5.6 [ox_reset ox_intr register_handler], page 110, Section 7.5.8
[ox_pop_cmo ox_pop_local], page 112.

8 Groebner basis computation

8.1 Distributed polynomial

A distributed polynomial is a polynomial with a special internal representation different from the ordinary one.

An ordinary polynomial (having `type` 2) is internally represented in a format, called recursive representation. In fact, it is represented as an uni-variate polynomial with respect to a fixed variable, called main variable of that polynomial, where the other variables appear in the coefficients which may again polynomials in such variables other than the previous main variable. A polynomial in the coefficients is again represented as an uni-variate polynomial in a certain fixed variable, the main variable. Thus, by this recursive structure of polynomial representation, it is called the 'recursive representation.'

$$(x + y + z)^2 = 1 \cdot x^2 + (2 \cdot y + (2 \cdot z)) \cdot x + ((2 \cdot z) \cdot y + (1 \cdot z^2))$$

On the other hand, we call a representation the distributed representation of a polynomial, if a polynomial is represented, according to its original meaning, as a sum of monomials, where a monomial is the product of power product of variables and a coefficient. We call a polynomial, represented in such an internal format, a distributed polynomial. (This naming may sounds something strange.)

$$(x + y + z)^2 = 1 \cdot x^2 + 2 \cdot xy + 2 \cdot xz + 1 \cdot y^2 + 2 \cdot yz + 1 \cdot z^2$$

For computation of Groebner basis, efficient operation is expected if polynomials are represented in a distributed representation, because major operations for Groebner basis are performed with respect to monomials. From this view point, we provide the object type distributed polynomial with its object identification number 9, and objects having such a type are available by **Asir** language.

Here, we provide several definitions for the later description.

term   The power product of variables, i.e., a monomial with coefficient 1. In an **Asir** session, it is displayed in the form like

      `<<0,1,2,3,4>>`

and also can be input in such a form. This example shows a term in 5 variables. If we assume the 5 variables as `a`, `b`, `c`, `d`, and `e`, the term represents `b*c^2*d^3*e^4` in the ordinary expression.

term order

      Terms are ordered according to a total order with the following properties.

    1. For all `t` `t > 1`.
    2. For all `t`, `s`, `u` `t > s` implies `tu > su`.

      Such a total order is called a term ordering. A term ordering is specified by a variable ordering (a list of variables) and a type of term ordering (an integer, a list or a matrix).

monomial   The product of a term and a coefficient. In an **Asir** session, it is displayed in the form like

      `2*<<0,1,2,3,4>>`

and also can be input in such a form.

head term
head monomial
head coefficient

> Monomials in a distributed polynomial is sorted by a total order. In such representation, we call the monomial that is maximum with respect to the order the head monomial, and its term and coefficient the head term and the head coefficient respectively.

ChangeLog

- Distributed polynomials accept objects as coefficients. The k-th element of a free module is expressed as `<<d0,d1,...:k>>` (2017-08-31).

- 1.15 algnum.c, 1.53 ctrl.c, 1.66 dp-supp.c, 1.105 dp.c, 1.73 gr.c, 1.4 reduct.c, 1.16 _distm.c, 1.17 dalg.c, 1.52 dist.c, 1.20 distm.c, 1.8 gmpq.c, 1.238 engine/nd.c, 1.102 ca.h, 1.411 version.h, 1.28 cpexpr.c, 1.42 pexpr.c, 1.20 pexpr_body.c, 1.40 spexpr.c, 1.27 arith.c, 1.77 eval.c, 1.56 parse.h, 1.37 parse.y, 1.8 stdio.c, 1.31 plotf.c

8.2 Fundamental functions

In the version before 2003, facilities for computing Groebner bases are `dp_gr_main()`, `dp_gr_mod_main()` and `dp_gr_f_main()`. To call these functions, it is necessary to set several parameters correctly and it is convenient to use a set of interface functions provided in the library file 'gr'. The facilities will be ready to use after you load the package by `load()`. The package 'gr' is placed in the standard library directory of **Asir**.

In the current version, new functions such as `nd_gr`, `nd_f4` are available and these function can compute Groebner bases more efficiently than old functions (Section 8.10.8 [nd_gr nd_gr_trace nd_f4 nd_f4_trace nd_weyl_gr nd_weyl_gr_trace], page 138).

There are many functions and options defined in the package 'gr'. Usually not so many of them are used. Top level functions for Groebner basis computation are the following three functions.

In the following description, *plist*, *vlist*, *order* and *p* stand for a list of polynomials, a list of variables (indeterminates), a type of term ordering and a prime less than `2^27` respectively.

`gr(`*plist*`,`*vlist*`,`*order*`)`

> Function that computes Groebner bases over the rationals. The algorithm is Buchberger algorithm with useless pair elimination criteria by Gebauer-Moeller, sugar strategy and trace-lifting by Traverso. For ordinary computation, this function is used.

`hgr(`*plist*`,`*vlist*`,`*order*`)`

> After homogenizing the input polynomials a candidate of the \gr basis is computed by trace-lifting. Then the candidate is dehomogenized and checked whether it is indeed a Groebner basis of the input. Sugar strategy often causes intermediate coefficient swells. It is empirically known that the combination of homogenization and suppresses the swells for such cases.

`gr_mod(`*plist*`,`*vlist*`,`*order*`,`*p*`)`

> Function that computes Groebner bases over $GF(p)$. The same algorithm as `gr()` is used.

In the current version, the functions corresponding to these three interfaces are provided by `nd_gr`, `nd_gr_trace`. See Section 8.10.8 [nd_gr nd_gr_trace nd_f4 nd_f4_trace nd_weyl_gr nd_weyl_gr_trace], page 138.

8.3 Controlling Groebner basis computations

One can control a Groebner basis computation by setting various parameters. These parameters can be set and examined by a built-in function `dp_gr_flags()`. Without argument it returns the current settings.

```
[100] dp_gr_flags();
[Demand,0,NoSugar,0,NoCriB,0,NoGC,0,NoMC,0,NoRA,0,NoGCD,0,Top,0,
ShowMag,1,Print,1,Stat,0,Reverse,0,InterReduce,0,Multiple,0]
[101]
```

The return value is a list which contains the names of parameters and their values. The meaning of the parameters are as follows. 'on' means that the parameter is not zero.

NoSugar   If 'on', Buchberger's normal strategy is used instead of sugar strategy.

NoCriB   If 'on', criterion B among the Gebauer-Moeller's criteria is not applied.

NoGC   If 'on', the check that a Groebner basis candidate is indeed a Groebner basis, is not executed.

NoMC   If 'on', the check that the resulting polynomials generates the same ideal as the ideal generated by the input, is not executed.

NoRA   If 'on', the interreduction, which makes the Groebner basis reduced, is not executed.

NoGCD   If 'on', content removals are not executed during a Groebner basis computation over a rational function field.

Top   If 'on', Only the head term of the polynomial being reduced is reduced.

Reverse   If 'on', the selection strategy of reducer in a normal form computation is such that a newer reducer is used first.

Print   If 'on', various information during a Groebner basis computation is displayed.

PrintShort
            If 'on' and Print is 'off', short information during a Groebner basis computation is displayed.

Stat   If 'on', a summary of information is shown after a Groebner basis computation. Note that the summary is always shown if `Print` is 'on'.

ShowMag   If 'on' and `Print` is 'on', the sum of bit length of coefficients of a generated basis element, which we call *magnitude*, is shown after every normal computation. After comleting the computation the maximal value among the sums is shown.

Content
Multiple   If a non-zero rational number, in a normal form computation over the rationals, the integer content of the polynomial being reduced is removed when its magnitude becomes `Content` times larger than a registered value, which is set

to the magnitude of the input polynomial. After each content removal the registered value is set to the magnitude of the resulting polynomial. `Content` is equal to 1, the simiplification is done after every normal form computation. It is empirically known that it is often efficient to set `Content` to 2 for the case where large integers appear during the computation. An integer value can be set by the keyword `Multiple` for backward compatibility.

Demand

If the value (a character string) is a valid directory name, then generated basis elements are put in the directory and are loaded on demand during normal form computations. Each elements is saved in the binary form and its name coincides with the index internally used in the computation. These binary files are not removed automatically and one should remove them by hand.

If `Print` is 'on', the following information are shown.

```
[93] gr(cyclic(4),[c0,c1,c2,c3],0)$
mod= 99999989, eval = []
(0)(0)<<0,2,0,0>>(2,3),nb=2,nab=5,rp=2,sugar=2,mag=4
(0)(0)<<0,1,2,0>>(1,2),nb=3,nab=6,rp=2,sugar=3,mag=4
(0)(0)<<0,1,1,2>>(0,1),nb=4,nab=7,rp=3,sugar=4,mag=6
.
(0)(0)<<0,0,3,2>>(5,6),nb=5,nab=8,rp=2,sugar=5,mag=4
(0)(0)<<0,1,0,4>>(4,6),nb=6,nab=9,rp=3,sugar=5,mag=4
(0)(0)<<0,0,2,4>>(6,8),nb=7,nab=10,rp=4,sugar=6,mag=6
....gb done
reduceall
.......
membercheck
(0,0)(0,0)(0,0)(0,0)
gbcheck total 8 pairs
........
UP=(0,0)SP=(0,0)SPM=(0,0)NF=(0,0)NFM=(0.010002,0)ZNFM=(0.010002,0)
PZ=(0,0)NP=(0,0)MP=(0,0)RA=(0,0)MC=(0,0)GC=(0,0)T=40,B=0 M=8 F=6
D=12 ZR=5 NZR=6 Max_mag=6
[94]
```

In this example `mod` and `eval` indicate moduli used in trace-lifting. `mod` is a prime and `eval` is a list of integers used for evaluation when the ground field is a field of rational functions.

The following information is shown after every normal form computation.

```
(TNF)(TCONT)HT(INDEX),nb=NB,nab=NAB,rp=RP,sugar=S,mag=M
```

Meaning of each component is as follows.

TNF

CPU time for normal form computation (second)

TCONT

CPU time for content removal(second)

HT

        Head term of the generated basis element

INDEX

        Pair of indices which corresponds to the reduced S-polynomial

NB

        Number of basis elements after removing redundancy

NAB

        Number of all the basis elements

RP

        Number of remaining pairs

S

        Sugar of the generated basis element

M

        Magnitude of the generated basis element (shown if `ShowMag` is 'on'.)

The summary of the information shown after a Groebner basis computation is as follows. If a component shows timings and it contains two numbers, they are a pair of time for computation and time for garbage collection.

UP

        Time to manipulate the list of critical pairs

SP

        Time to compute S-polynomials over the rationals

SPM

        Time to compute S-polynomials over a finite field

NF

        Time to compute normal forms over the rationals

NFM

        Time to compute normal forms over a finite field

ZNFM

        Time for zero reductions in `NFM`

PZ

        Time to remove integer contents

NP

        Time to compute remainders for coefficients of polynomials with coefficients in the rationals

MP

        Time to select pairs from which S-polynomials are computed

RA

Time to interreduce the Groebner basis candidate

MC

Time to check that each input polynomial is a member of the ideal generated by the Groebner basis candidate.

GC

Time to check that the Groebner basis candidate is a Groebner basis

T

Number of critical pairs generated

B, M, F, D

Number of critical pairs removed by using each criterion

ZR

Number of S-polynomials reduced to 0

NZR

Number of S-polynomials reduced to non-zero results

Max_mag

Maximal magnitude among all the generated polynomials

8.4 Setting term orderings

A term is internally represented as an integer vector whose components are exponents with respect to variables. A variable list specifies the correspondences between variables and components. A type of term ordering specifies a total order for integer vectors. A type of term ordering is represented by an integer, a list of integer or matrices.

There are following three fundamental types.

0 (DegRevLex; **total degree reverse lexicographic ordering**)

In general, computation by this ordering shows the fastest speed in most Groebner basis computations. However, for the purpose to solve polynomial equations, this type of ordering is, in general, not so suitable. The Groebner bases obtained by this ordering is used for computing the number of solutions, solving ideal membership problem and seeds for conversion to other Groebner bases under different ordering.

1 (DegLex; **total degree lexicographic ordering**)

By this type term ordering, Groebner bases are obtained fairly faster than Lex (lexicographic) ordering, too. Alike the DegRevLex ordering, the result, in general, cannot directly be used for solving polynomial equations. It is used, however, in such a way that a Groebner basis is computed in this ordering after homogenization to obtain the final lexicographic Groebner basis.

2 (Lex; **lexicographic ordering**)

Groebner bases computed by this ordering give the most convenient Groebner bases for solving the polynomial equations. The only and serious shortcoming

is the enormously long computation time. It is often observed that the number coefficients of the result becomes very very long integers, especially if the ideal is 0-dimensional. For such a case, it is empirically true for many cases that i.e., computation by `gr()` and/or `hgr()` may be quite effective.

By combining these fundamental orderingl into a list, one can make various term ordering called elimination orderings.

```
[[O1,L1],[O2,L2],...]
```

In this example `Oi` indicates 0, 1 or 2 and `Li` indicates the number of variables subject to the corresponding orderings. This specification means the following.

The variable list is separated into sub lists from left to right where the `i`-th list contains `Li` members and it corresponds to the ordering of type `Oi`. The result of a comparison is equal to that for the leftmost different sub components. This type of ordering is called an elimination ordering.

Furthermore one can specify a term ordering by a matrix. Suppose that a real `n`, `m` matrix `M` has the following properties.

1. For all integer vectors `v` of length `m` `Mv=0` is equivalent to `v=0`.

2. For all non-negative integer vectors `v` the first non-zero component of `Mv` is non-negative.

Then we can define a term ordering such that, for two vectors `t`, `s`, `t>s` means that the first non-zero component of `M(t-s)` is non-negative.

Types of term orderings are used as arguments of functions such as `gr()`. It is also set internally by `dp_ord()` and is used during executions of various functions.

For concrete definitions of term ordering and more information about Groebner basis, refer to, for example, the book `[Becker,Weispfenning]`.

Note that the variable ordering have strong effects on the computation time as well as the choice of types of term orderings.

```
[90] B=[x^10-t,x^8-z,x^31-x^6-x-y]$
[91] gr(B,[x,y,z,t],2);
[x^2-2*y^7+(-41*t^2-13*t-1)*y^2+(2*t^17-12*t^14+42*t^12+30*t^11-168*t^9
-40*t^8+70*t^7+252*t^6+30*t^5-140*t^4-168*t^3+2*t^2-12*t+16)*z^2*y
+(-12*t^16+72*t^13-28*t^11-180*t^10+112*t^8+240*t^7+28*t^6-127*t^5
-167*t^4-55*t^3+30*t^2+58*t-15)*z^4,
(y+t^2*z^2)*x+y^7+(20*t^2+6*t+1)*y^2+(-t^17+6*t^14-21*t^12-15*t^11
+84*t^9+20*t^8-35*t^7-126*t^6-15*t^5+70*t^4+84*t^3-t^2+5*t-9)*z^2*y
+(6*t^16-36*t^13+14*t^11+90*t^10-56*t^8-120*t^7-14*t^6+64*t^5+84*t^4
+27*t^3-16*t^2-30*t+7)*z^4,
(t^3-1)*x-y^6+(-6*t^13+24*t^10-20*t^8-36*t^7+40*t^5+24*t^4-6*t^3-20*t^2
-6*t-1)*y+(t^17-6*t^14+9*t^12+15*t^11-36*t^9-20*t^8-5*t^7+54*t^6+15*t^5
+10*t^4-36*t^3-11*t^2-5*t+9)*z^2,
-y^8-8*t*y^3+16*z^2*y^2+(-8*t^16+48*t^13-56*t^11-120*t^10+224*t^8+160*t^7
-56*t^6-336*t^5-112*t^4+112*t^3+224*t^2+24*t-56)*z^4*y+(t^24-8*t^21
+20*t^19+28*t^18-120*t^16-56*t^15+14*t^14+300*t^13+70*t^12-56*t^11
-400*t^10-84*t^9+84*t^8+268*t^7+84*t^6-56*t^5-63*t^4-36*t^3+46*t^2
-12*t+1)*z,2*t*y^5+z*y^2+(-2*t^11+8*t^8-20*t^6-12*t^5+40*t^3+8*t^2
-10*t-20)*z^3*y+8*t^14-32*t^11+48*t^8-t^7-32*t^5-6*t^4+9*t^2-t,
```

```
    -z*y^3+(t^7-2*t^4+3*t^2+t)*y+(-2*t^6+4*t^3+2*t-2)*z^2,
    2*t^2*y^3+z^2*y^2+(-2*t^5+4*t^2-6)*z^4*y
    +(4*t^8-t^7-8*t^5+2*t^4-4*t^3+5*t^2-t)*z,
    z^3*y^2+2*t^3*y+(-t^7+2*t^4+t^2-t)*z^2,
    -t*z*y^2-2*z^3*y+t^8-2*t^5-t^3+t^2,
    -t^3*y^2-2*t^2*z^2*y+(t^6-2*t^3-t+1)*z^4,z^5-t^4]
    [93] gr(B,[t,z,y,x],2);
    [x^10-t,x^8-z,x^31-x^6-x-y]
```

As you see in the above example, the Groebner base under variable ordering `[x,y,z,t]` has a lot of bases and each base itself is large. Under variable ordering `[t,z,y,x]`, however, `B` itself is already the Groebner basis. Roughly speaking, to obtain a Groebner base under the lexicographic ordering is to express the variables on the left (having higher order) in terms of variables on the right (having lower order). In the example, variables `t`, `z`, and `y` are already expressed by variable `x`, and the above explanation justifies such a drastic experimental results. In practice, however, optimum ordering for variables may not known beforehand, and some heuristic trial may be inevitable.

8.5 Weight

Term orderings introduced in the previous section can be generalized by setting a weight for each variable.

```
    [0] dp_td(<<1,1,1>>);
    3
    [1] dp_set_weight([1,2,3])$
    [2] dp_td(<<1,1,1>>);
    6
```

By default, the total degree of a monomial is equal to the sum of all exponents. This means that the weight for each variable is set to 1. In this example, the weights for the first, the second and the third variable are set to 1, 2 and 3 respectively. Therefore the total degree of `<<1,1,1>>` under this weight, which is called the weight of the monomial, is `1*1+1*2+1*3=6`. By setting weights, different term orderings can be set under a type of term ordeing. In some case a polynomial can be made weighted homogeneous by setting an appropriate weight.

A list of weights for all variables is called a weight vector. A weight vector is called a sugar weight vector if its elements are all positive and it is used for computing a weighted total degree of a monomial, because such a weight is used instead of total degree in sugar strategy. On the other hand, a weight vector whose elements are not necessarily positive cannot be set as a sugar weight, but it is useful for generalizing term order. In fact, such a weight vector already appeared in a matrix order. That is, each row of a matrix defining a term order is regarded as a weight vector. A block order is also considered as a refinement of comparison by weight vectors. It compares two terms by using a weight vector whose elements corresponding to variables in a block is 1 and 0 otherwise, then it applies a tie breaker.

A weight vector can be set by using `dp_set_weight()`. However it is more preferable if a weight vector can be set together with other parapmeters such as a type of term ordering and a variable order. This is realized as follows.

```
[64] B=[x+y+z-6,x*y+y*z+z*x-11,x*y*z-6]$
[65] dp_gr_main(B|v=[x,y,z],sugarweight=[3,2,1],order=0);
[z^3-6*z^2+11*z-6,x+y+z-6,-y^2+(-z+6)*y-z^2+6*z-11]
[66] dp_gr_main(B|v=[y,z,x],order=[[1,1,0],[0,1,0],[0,0,1]]);
[x^3-6*x^2+11*x-6,x+y+z-6,-x^2+(-y+6)*x-y^2+6*y-11]
[67] dp_gr_main(B|v=[y,z,x],order=[[x,1,y,2,z,3]]);
[x+y+z-6,x^3-6*x^2+11*x-6,-x^2+(-y+6)*x-y^2+6*y-11]
```

In each example, a term ordering is specified as options. In the first example, a variable order, a sugar weight vector and a type of term ordering are specified by options v, sugarweight and order respectively. In the second example, an option order is used to set a matrix ordering. That is, the specified weight vectors are used from left to right for comparing terms. The third example shows a variant of specifying a weight vector, where each component of a weight vector is specified variable by variable, and unspecified components are set to zero. In this example, a term order is not determined only by the specified weight vector. In such a case a tie breaker by the graded reverse lexicographic ordering is set automatically. This type of a term ordering specification can be applied only to builtin functions such as dp_gr_main(), dp_gr_mod_main(), not to user defined functions such as gr().

8.6 Groebner basis computation with rational function coefficients

Such variables that appear within the input polynomials but not appearing in the input variable list are automatically treated as elements in the coefficient field by top level functions, such as gr().

```
[64] gr([a*x+b*y-c,d*x+e*y-f],[x,y],2);
[(-e*a+d*b)*x-f*b+e*c,(-e*a+d*b)*y+f*a-d*c]
```

In this example, variables a, b, c, and d are treated as elements in the coefficient field. In this case, a Groebner basis is computed on a bi-variate polynomial ring $\mathbf{F}[\mathbf{x},\mathbf{y}]$ over rational function field $\mathbf{F} = \mathbf{Q}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$. Notice that coefficients are considered as a member in a field. As a consequence, polynomial factors common to the coefficients are removed so that the result, in general, is different from the result that would be obtained when the problem is considered as a computation of Groebner basis over a polynomial ring with rational function coefficients. And note that coefficients of a distributed polynomial are limited to numbers and polynomials because of efficiency.

8.7 Change of orderng

When we compute a lex order Groebner basis, it is often efficient to compute it via Groebner basis with respect to another order such as degree reverse lex order, rather than to compute it directory by gr() etc. If we know that an input is a Groebner basis with respect to an order, we can apply special methods called change of ordering for a Groebner basis computation with respect to another order, without using Buchberger algorithm. The following two functions are ones for change of ordering such that they convert a Groebner basis *gbase* with respect to the variable order *vlist1* and the order type *order* into a lex Groebner basis with respect to the variable order *vlist2*.

```
tolex(gbase,vlist1,order,vlist2)
```
> This function can be used only when *gbase* is an ideal over the rationals. The input *gbase* must be a Groebner basis with respect to the variable order *vlist1* and the order type *order*. Moreover the ideal generated by *gbase* must be zero-dimensional. This computes the lex Groebner basis of *gbase* by using the modular change of ordering algorithm. The algorithm first computes the lex Groebner basis over a finite field. Then each element in the lex Groebner basis over the rationals is computed with undetermined coefficient method and linear equation solving by Hensel lifting.

```
tolex_tl(gbase,vlist1,order,vlist2,homo)
```
> This function computes the lex Groebner basis of *gbase*. The input *gbase* must be a Groebner basis with respect to the variable order *vlist1* and the order type *order*. Buchberger algorithm with trace lifting is used to compute the lex Groebner basis, however the Groebner basis check and the ideal membership check can be omitted by using several properties derived from the fact that the input is a Groebner basis. So it is more efficient than simple repetition of Buchberger algorithm. If the input is zero-dimensional, this function inserts automatically a computation of Groebner basis with respect to an elimination order, which makes the whole computation more efficient for many cases. If *homo* is not equal to 0, homogenization is used in each step.

For zero-dimensional systems, there are several functions to compute the minimal polynomial of a polynomial and or a more compact representation for zeros of the system. They are all defined in 'gr'. Refer to the sections for each functions.

8.8 Weyl algebra

So far we have explained Groebner basis computation in commutative polynomial rings. However Groebner basis can be considered in more general non-commutative rings. Weyl algebra is one of such rings and Risa/Asir implements fundamental operations in Weyl algebra and Groebner basis computation in Weyl algebra.

The `n` dimensional Weyl algebra over a field K, `D=K<x1,...,xn,D1,...,Dn>` is a non-commutative algebra which has the following fundamental relations:

```
xi*xj-xj*xi=0, Di*Dj-Dj*Di=0, Di*xj-xj*Di=0 (i!=j), Di*xi-xi*Di=1
```

`D` is the ring of differential operators whose coefficients are polynomials in `K[x1,...,xn]` and `Di` denotes the differentiation with respect to `xi`. According to the commutation relation, elements of `D` can be represented as a K-linear combination of monomials `x1^i1*...*xn^in*D1^j1*...*Dn^jn`. In Risa/Asir, this type of monomial is represented by `<<i1,...,in,j1,...,jn>>` as in the case of commutative polynomial. That is, elements of `D` are represented by distributed polynomials. Addition and subtraction can be done by `+`, `-`, but multiplication is done by calling `dp_weyl_mul()` because of the non-commutativity of `D`.

```
[0] A=<<1,2,2,1>>;
(1)*<<1,2,2,1>>
[1] B=<<2,1,1,2>>;
(1)*<<2,1,1,2>>
```

```
[2] A*B;
(1)*<<3,3,3,3>>
[3] dp_weyl_mul(A,B);
(1)*<<3,3,3,3>>+(1)*<<3,2,3,2>>+(4)*<<2,3,2,3>>+(4)*<<2,2,2,2>>
+(2)*<<1,3,1,3>>+(2)*<<1,2,1,2>>
```

The following functions are available for Groebner basis computation in Weyl algebra: `dp_weyl_gr_main()`, `dp_weyl_gr_mod_main()`, `dp_weyl_gr_f_main()`, `dp_weyl_f4_main()`, `dp_weyl_f4_mod_main()`. Computation of the global b function is implemented as an application.

8.9 Module over a polynomial ring

not yet

8.10 Functions for Groebner basis computation

8.10.1 `gr, hgr, gr_mod, dgr`

`gr(`*plist,vlist,order*`)`
`hgr(`*plist,vlist,order*`)`
`gr_mod(`*plist,vlist,order,p*`)`
`dgr(`*plist,vlist,order,procs*`)`
    :: Groebner basis computation

return   list

plist vlist procs
    list

order   number, list or matrix

p   prime less than 2^27

- These functions are defined in '`gr`' in the standard library directory.
- Functions of which names contains gr are obsoleted. Functions of `nd_gr` families should be used (Section 8.10.8 [`nd_gr nd_gr_trace nd_f4 nd_f4_trace nd_weyl_gr nd_weyl_gr_trace`], page 138).
- They compute a Groebner basis of a polynomial list *plist* with respect to the variable order *vlist* and the order type *order*. `gr()` and `hgr()` compute a Groebner basis over the rationals and `gr_mod` computes over GF(p).
- Variables not included in *vlist* are regarded as included in the ground field.
- `gr()` uses trace-lifting (an improvement by modular computation) and sugar strategy. `hgr()` uses trace-lifting and a cured sugar strategy by using homogenization.
- `dgr()` executes `gr()`, `dgr()` simultaneously on two process in a child process list *procs* and returns the result obtained first. The results returned from both the process should be equal, but it is not known in advance which method is faster. Therefore this function is useful to reduce the actual elapsed time.
- The CPU time shown after an execution of `dgr()` indicates that of the master process, and most of the time corresponds to the time for communication.

- When the elements of *plist* are distributed polynomials, the result is also a list of distributed polynomials. In this case, firstly the elements of *plist* is sorted by `dp_sort` and the Grobner basis computation is started. Variables must be given in *vlist* even in this case (these variables are dummy).

```
[0] load("gr")$
[64] load("cyclic")$
[74] G=gr(cyclic(5),[c0,c1,c2,c3,c4],2);
[c4^15+122*c4^10-122*c4^5-1,...]
[75] GM=gr_mod(cyclic(5),[c0,c1,c2,c3,c4],2,31991)$
24628*c4^15+29453*c4^10+2538*c4^5+7363
[76] (G[0]*24628-GM[0])%31991;
0
```

References
>   Section 8.10.6 [dp_gr_main dp_gr_mod_main dp_gr_f_main dp_weyl_gr_main
>   dp_weyl_gr_mod_main dp_weyl_gr_f_main], page 136, Section 8.10.11 [dp_
>   ord], page 140.

8.10.2 lex_hensel, lex_tl, tolex, tolex_d, tolex_tl

lex_hensel(*plist*,*vlist1*,*order*,*vlist2*,*homo*)
lex_tl(*plist*,*vlist1*,*order*,*vlist2*,*homo*)
>   : Groebner basis computation with respect to a lex order by change of ordering

tolex(*plist*,*vlist1*,*order*,*vlist2*)
tolex_d(*plist*,*vlist1*,*order*,*vlist2*,*procs*)
tolex_tl(*plist*,*vlist1*,*order*,*vlist2*,*homo*)
>   :: Groebner basis computation with respect to a lex order by change of ordering,
>   starting from a Groebner basis

return   list

plist vlist1 vlist2 procs
>   list

order   number, list or matrix

homo   flag

- These functions are defined in '`gr`' in the standard library directory.
- `lex_hensel()` and `lex_tl()` first compute a Groebner basis with respect to the variable order *vlist1* and the order type *order*. Then the Groebner basis is converted into a lex order Groebner basis with respect to the variable order *vlist2*.
- `tolex()` and `tolex_tl()` convert a Groebner basis *plist* with respect to the variable order *vlist1* and the order type *order* into a lex order Groebner basis with respect to the variable order *vlist2*. `tolex_d()` does computations of basis elements in `tolex()` in parallel on the processes in a child process list *procs*.
- In `lex_hensel()` and `tolex_hensel()` a lex order Groebner basis is computed as follows.(Refer to [Noro,Yokoyama].)
  1. Compute a Groebner basis *G0* with respect to *vlist1* and *order*. (Only in `lex_hensel()`.)

2. Choose a prime which does not divide head coefficients of elements in *G0* with respect to *vlist1* and *order*. Then compute a lex order Groebner basis *Gp* over GF(p) with respect to *vlist2*.

3. Compute *NF*, the set of all the normal forms with respect to *G0* of terms appearing in *Gp*.

4. For each element f in *Gp*, replace coefficients and terms in f with undetermined coefficients and the corresponding polynomials in *NF* respectively, and generate a system of liear equation *Lf* by equating the coefficients of terms in the replaced polynomial with 0.

5. Solve *Lf* by Hensel lifting, starting from the unique mod p solution.

6. If all the linear equations generated from the elements in *Gp* could be solved, then the set of solutions corresponds to a lex order Groebner basis. Otherwise redo the whole process with another p.

- In `lex_tl()` and `tolex_tl()` a lex order Groebner basis is computed as follows.(Refer to `[Noro,Yokoyama]`.)

  1. Compute a Groebner basis *G0* with respect to *vlist1* and *order*. (Only in `lex_tl()`.)

  2. If *G0* is not zero-dimensional, choose a prime which does not divide head coefficients of elements in *G0* with respect to *vlist1* and *order*. Then compute a candidate of a lex order Groebner basis via trace lifting with p. If it succeeds the candidate is indeed a lex order Groebner basis without any check. Otherwise redo the whole process with another p.

  3.

     If *G0* is zero-dimensional, starting from *G0*, compute a Groebner basis *G1* with respect to an elimination order to eliminate variables other than the last variable in *vlist2*. Then compute a lex order Groebner basis stating from *G1*. These computations are done by trace lifting and the selection of a mudulus p is the same as in non zero-dimensional cases.

- Computations with rational function coefficients can be done only by `lex_tl()` and `tolex_tl()`.

- If `homo` is not equal to 0, homogenization is used in Buchberger algorithm.

- The CPU time shown after an execution of `tolex_d()` indicates that of the master process, and it does not include the time in child processes.

```
[78] K=katsura(5)$
30msec + gc : 20msec
[79] V=[u5,u4,u3,u2,u1,u0]$
0msec
[80] G0=hgr(K,V,2)$
91.558sec + gc : 15.583sec
[81] G1=lex_hensel(K,V,0,V,0)$
49.049sec + gc : 9.961sec
[82] G2=lex_tl(K,V,0,V,1)$
31.186sec + gc : 3.500sec
[83] gb_comp(G0,G1);
```

```
    1
    10msec
    [84] gb_comp(G0,G2);
    1
```

References
        Section 8.10.6 [`dp_gr_main dp_gr_mod_main dp_gr_f_main dp_weyl_gr_main dp_weyl_gr_mod_main dp_weyl_gr_f_main`], page 136, Section 8.10.11 [`dp_ord`], page 140, Chapter 7 [`Distributed computation`], page 102

8.10.3 `lex_hensel_gsl`, `tolex_gsl`, `tolex_gsl_d`

`lex_hensel_gsl(`*plist*`,`*vlist1*`,`*order*`,`*vlist2*`,`*homo*`)`
        ::Computation of an GSL form ideal basis

`tolex_gsl(`*plist*`,`*vlist1*`,`*order*`,`*vlist2*`)`
`tolex_gsl_d(`*plist*`,`*vlist1*`,`*order*`,`*vlist2*`,`*procs*`)`
        :: Computation of an GSL form ideal basis stating from a Groebner basis

return   list

plist vlist1 vlist2 procs
        list

order   number, list or matrix

homo   flag

- `lex_hensel_gsl()` and `lex_hensel()` are variants of `tolex_gsl()` and `tolex()` respectively. The results are Groebner basis or a kind of ideal basis, called GSL form. `tolex_gsl_d()` does basis computations in parallel on child processes specified in `procs`.

- If the input is zero-dimensional and a lex order Groebner basis has the form `[f0,x1-f1,...,xn-fn]` (`f0,...,fn` are univariate polynomials of `x0`; SL form), then this these functions return a list such as `[[x1,g1,d1],...,[xn,gn,dn],[x0,f0,f0']]` (GSL form). In this list `gi` is a univariate polynomial of `x0` such that `di*f0'*fi-gi` divides `f0` and the roots of the input ideal is `[x1=g1/(d1*f0'),...,xn=gn/(dn*f0')]` for `x0` such that `f0(x0)=0`. If the lex order Groebner basis does not have the above form, these functions return a lex order Groebner basis computed by `tolex()`.

- Though an ideal basis represented as GSL form is not a Groebner basis we can expect that the coefficients are much smaller than those in a Groebner basis and that the computation is efficient. The CPU time shown after an execution of `tolex_gsl_d()` indicates that of the master process, and it does not include the time in child processes.

```
    [103] K=katsura(5)$
    [104] V=[u5,u4,u3,u2,u1,u0]$
    [105] G0=gr(K,V,0)$
    [106] GSL=tolex_gsl(G0,V,0,V)$
    [107] GSL[0];
    [u1,86358374211304776672000000000*u0^31-...]
    [108] GSL[1];
    [u2,10352277157007342793600000000*u0^31-...]
```

```
[109] GSL[5];
[u0,11771021876193064124640000000*u0^32-...,
376672700038178051988480000000*u0^31-...]
```

```
References
```
          Section 8.10.2 [lex_hensel lex_tl tolex tolex_d tolex_tl], page 132,
          Chapter 7 [Distributed computation], page 102

8.10.4 gr_minipoly, minipoly

```
gr_minipoly(plist,vlist,order,poly,v,homo)
```
          :: Computation of the minimal polynomial of a polynomial modulo an ideal

```
minipoly(plist,vlist,order,poly,v)
```
          :: Computation of the minimal polynomial of a polynomial modulo an ideal

return   polynomial

plist vlist   list

order   number, list or matrix

poly   polynomial

v   indeterminate

homo   flag

- `gr_minipoly()` begins by computing a Groebner basis. `minipoly()` regards an input as a Groebner basis with respect to the variable order *vlist* and the order type *order*.

- Let K be a field. If an ideal I in K[X] is zero-dimensional, then, for a polynomial p in K[X], the kernel of a homomorphism from K[v] to K[X]/I which maps f(v) to f(p) mod I is generated by a polynomial. The generator is called the minimal polynomial of p modulo I.

- `gr_minipoly()` and `minipoly()` computes the minimal polynomial of a polynomial p and returns it as a polynomial of *v*.

- The minimal polynomial can be computed as an element of a Groebner basis. But if we are only interested in the minimal polynomial, `minipoly()` and `gr_minipoly()` can compute it more efficiently than methods using Groebner basis computation.

- It is recommended to use a degree reverse lex order as a term order for `gr_minipoly()`.

```
[117] G=tolex(G0,V,0,V)$
43.818sec + gc : 11.202sec
[118] GSL=tolex_gsl(G0,V,0,V)$
17.123sec + gc : 2.590sec
[119] MP=minipoly(G0,V,0,u0,z)$
4.370sec + gc : 780msec
```

```
References
```
          Section 8.10.2 [lex_hensel lex_tl tolex tolex_d tolex_tl], page 132.

8.10.5 `tolexm, minipolym`

`tolexm(`*plist*`,`*vlist1*`,`*order*`,`*vlist2*`,`*mod*`)`
:: Groebner basis computation modulo *mod* by change of ordering.

`minipolym(`*plist*`,`*vlist1*`,`*order*`,`*poly*`,`*v*`,`*mod*`)`
:: Minimal polynomial computation modulo *mod* the same method as

return   `tolexm()` : list, `minipolym()` : polynomial

plist vlist1 vlist2
list

order   number, list or matrix

mod   prime

- An input *plist* must be a Groebner basis modulo *mod* with respect to the variable order *vlist1* and the order type *order*.
- `minipolym()` executes the same computation as in `minipoly`.
- `tolexm()` computes a lex order Groebner basis modulo *mod* with respect to the variable order *vlist2*, by using FGLM algorithm.

```
[197] tolexm(G0,V,0,V,31991);
[8271*u0^31+10435*u0^30+816*u0^29+26809*u0^28+...,...]
[198] minipolym(G0,V,0,u0,z,31991);
z^32+11405*z^31+20868*z^30+21602*z^29+...
```

`References`
Section 8.10.2 [`lex_hensel lex_tl tolex tolex_d tolex_tl`], page 132, Section 8.10.4 [`gr_minipoly minipoly`], page 135.

8.10.6 `dp_gr_main, dp_gr_mod_main, dp_gr_f_main, dp_weyl_gr_main,` `dp_weyl_gr_mod_main, dp_weyl_gr_f_main`

`dp_gr_main(`*plist*`,`*vlist*`,`*homo*`,`*modular*`,`*order*`)`
`dp_gr_mod_main(`*plist*`,`*vlist*`,`*homo*`,`*modular*`,`*order*`)`
`dp_gr_f_main(`*plist*`,`*vlist*`,`*homo*`,`*order*`)`
`dp_weyl_gr_main(`*plist*`,`*vlist*`,`*homo*`,`*modular*`,`*order*`)`
`dp_weyl_gr_mod_main(`*plist*`,`*vlist*`,`*homo*`,`*modular*`,`*order*`)`
`dp_weyl_gr_f_main(`*plist*`,`*vlist*`,`*homo*`,`*order*`)`
:: Groebner basis computation (built-in functions)

return   list

plist vlist   list

order   number, list or matrix

homo   flag

modular   flag or prime

- These functions are fundamental built-in functions for Groebner basis computation and `gr()`,`hgr()` and `gr_mod()` are all interfaces to these functions. Functions whose names contain weyl are those for computation in Weyl algebra.

- `dp_gr_f_main()` and `dp_weyl_gr_f_main()` are functions for Groebner basis computation over various finite fields. Coefficients of input polynomials must be converted to elements of a finite field currently specified by `setmod_ff()`.

- If *homo* is not equal to 0, homogenization is applied before entering Buchberger algorithm

- For `dp_gr_mod_main()`, *modular* means a computation over GF(*modular*). For `dp_gr_main()`, *modular* has the following mean.

   1. If *modular* is 1 , trace lifting is used. Primes for trace lifting are generated by `lprime()`, starting from `lprime(0)`, until the computation succeeds.

   2. If *modular* is an integer greater than 1, the integer is regarded as a prime and trace lifting is executed by using the prime. If the computation fails then 0 is returned.

   3. If *modular* is negative, the above rule is applied for *-modular* but the Groebner basis check and ideal-membership check are omitted in the last stage of trace lifting.

- `gr(P,V,O)`, `hgr(P,V,O)` and `gr_mod(P,V,O,M)` execute `dp_gr_main(P,V,0,1,O)`, `dp_gr_main(P,V,1,1,O)` and `dp_gr_mod_main(P,V,0,M,O)` respectively.

- Actual computation is controlled by various parameters set by `dp_gr_flags()`, other then by *homo* and *modular*.

References

Section 8.10.11 [`dp_ord`], page 140, Section 8.10.10 [`dp_gr_flags dp_gr_print`], page 140, Section 8.10.1 [`gr hgr gr_mod`], page 131, Section 10.5.1 [`setmod_ff`], page 178, Section 8.3 [`Controlling Groebner basis computations`], page 123

8.10.7 `dp_f4_main, dp_f4_mod_main, dp_weyl_f4_main, dp_weyl_f4_ mod_main`

```
dp_f4_main(plist,vlist,order)
dp_f4_mod_main(plist,vlist,order)
dp_weyl_f4_main(plist,vlist,order)
dp_weyl_f4_mod_main(plist,vlist,order)
```
         :: Groebner basis computation by F4 algorithm (built-in functions)

return   list

plist vlist   list

order   number, list or matrix

- These functions compute Groebner bases by F4 algorithm.

- F4 is a new generation algorithm for Groebner basis computation invented by J.C. Faugere. The current implementation of `dp_f4_main()` uses Chinese Remainder theorem and not highly optimized.

- Arguments and actions are the same as those of `dp_gr_main()`, `dp_gr_mod_main()`, `dp_weyl_gr_main()`, `dp_weyl_gr_mod_main()`, except for lack of the argument for controlling homogenization.

References

8.10.8 `nd_gr`, `nd_gr_trace`, `nd_f4`, `nd_f4_trace`, `nd_weyl_gr`, `nd_weyl_gr_trace`

`nd_gr(`*plist*`,`*vlist*`,`*p*`,`*order*`[|`*option=value*`,...])`
`nd_gr_trace(`*plist*`,`*vlist*`,`*homo*`,`*p*`,`*order*`[|`*option=value*`,...])`
`nd_f4(`*plist*`,`*vlist*`,`*modular*`,`*order*`[|`*option=value*`,...])`
`nd_f4_trace(`*plist*`,`*vlist*`,`*homo*`,`*p*`,`*order*`[|`*option=value*`,...])`
`nd_weyl_gr(`*plist*`,`*vlist*`,`*p*`,`*order*`[|`*option=value*`,...])`
`nd_weyl_gr_trace(`*plist*`,`*vlist*`,`*homo*`,`*p*`,`*order*`[|`*option=value*`,...])`
:: Groebner basis computation (built-in functions)

return   list

plist vlist   list

order   number, list or matrix

homo   flag

modular   flag or prime

- These functions are new implementations for computing Groebner bases.

- `nd_gr` executes Buchberger algorithm over the rationals if `p` is 0, and that over GF(p) if `p` is a prime.

- `nd_gr_trace` executes the trace algorithm over the rationals. If `p` is 0 or 1, the trace algorithm is executed until it succeeds by using automatically chosen primes. If `p` a positive prime, the trace is computed over GF(p). If the trace algorithm fails 0 is returned. If `p` is negative, the Groebner basis check and ideal-membership check are omitted. In this case, an automatically chosen prime if `p` is 1, otherwise the specified prime is used to compute a Groebner basis candidate. Execution of `nd_f4_trace` is done as follows: For each total degree, an F4-reduction of S-polynomials over a finite field is done, and S-polynomials which give non-zero basis elements are gathered. Then F4-reduction over Q is done for the gathered S-polynomials. The obtained polynomial set is a Groebner basis candidate and the same check procedure as in the case of `nd_gr_trace` is done.

- `nd_f4` executes F4 algorithm over Q if `modular` is equal to 0, or over a finite field GF(`modular`) if `modular` is a prime number of machine size (`<2^29`). If *plist* is a list of polynomials, then a Groebner basis of the ideal generated by *plist* is computed. If *plist* is a list of lists of polynomials, then each list of polynomials are regarded as an element of a free module over a polynomial ring and a Groebner basis of the sub-module generated by *plist* in the free module. In the latter case a term order in the free module should be specified. This is specified by *[s,ord]*. If *s* is 0 then it means TOP (Term Over Position). If *s* is 1 then it means POT 1 (Position Over Term). *ord* is a term order in the base polynomial ring.

- `nd_weyl_gr`, `nd_weyl_gr_trace` are for Weyl algebra computation.

- Functions except for F4 related ones can handle rational coefficient cases.
- In general these functions are more efficient than `dp_gr_main`, `dp_gr_mod_main`, especially over finite fields.
- The following options can be specified.

  homo   If set to 1, the computation is done via homogenization. (only for `nd_gr` and `nd_f4`)

  dp   If set to 1, the functions return a list of distributed polynomials (a list of module polynomials when the input is a sub-module).

  nora   If set to 1, the inter-reduction is not performed.

```
[38] load("cyclic")$
[49] C=cyclic(7)$
[50] V=vars(C)$
[51] cputime(1)$
[52] dp_gr_mod_main(C,V,0,31991,0)$
26.06sec + gc : 0.313sec(26.4sec)
[53] nd_gr(C,V,31991,0)$
ndv_alloc=1477188
5.737sec + gc : 0.1837sec(5.921sec)
[54] dp_f4_mod_main(C,V,31991,0)$
3.51sec + gc : 0.7109sec(4.221sec)
[55] nd_f4(C,V,31991,0)$
1.906sec + gc : 0.126sec(2.032sec)
```

References
> Section 8.10.11 [dp_ord], page 140, Section 8.10.10 [dp_gr_flags dp_gr_print], page 140, Section 8.3 [Controlling Groebner basis computations], page 123

8.10.9 nd_gr_postproc, nd_weyl_gr_postproc

nd_gr_postproc(*plist*,*vlist*,*p*,*order*,*check*)
nd_weyl_gr_postproc(*plist*,*vlist*,*p*,*order*,*check*)
> :: Check of Groebner basis candidate and inter-reduction

return   list or 0

plist vlist   list

p   prime or 0

order   number, list or matrix

check   0 or 1

- Perform the inter-reduction for a Groebner basis (candidate).
- `nd_weyl_gr_postproc` is for Weyl algebra.
- If *check=1* then the check whether *plist* is a Groebner basis with respect to a term order in a polynomial ring or Weyl algebra specified by *vlist*, *p* and *order*.

- This function is used for inter-reduction of a non-reduced Groebner basis that is obtained by dehomogenizing a Groebner basis computed via homogenization, or Groebner basis check of a Groebner basis candidate computed by CRT.

      afo

8.10.10 `dp_gr_flags`, `dp_gr_print`

`dp_gr_flags([`*list*`])`
`dp_gr_print([`*i*`])`
          and showing information.

return   value currently set

list   list

i   integer

- `dp_gr_flags()` sets and shows various parameters for Groebner basis computation.
- If no argument is specified the current settings are returned.
- Arguments must be specified as a list such as `["Print",1,"NoSugar",1,...]`. Names of parameters must be character strings.
- `dp_gr_print()` is used to set and show the value of a parameter `Print` and `PrintShort`.

    i=0   `Print=0, PrintShort=0`

    i=1   `Print=1, PrintShort=0`

    i=2   `Print=0, PrintShort=1`

    This functions is prepared to get quickly the value when a user defined function calling `dp_gr_main()` etc. uses the value as a flag for showing intermediate information.

References
          Section 8.3 [`Controlling Groebner basis computations`], page 123

8.10.11 `dp_ord`

`dp_ord([`*order*`])`
          :: Set and show the ordering type.

return   ordering type (number, list or matrix)

order   number, list or matrix

- If an argument is specified, the function sets the current ordering type to *order*. If no argument is specified, the function returns the ordering type currently set.
- There are two types of functions concerning distributed polynomial, functions which take a ordering type and those which don't take it. The latter ones use the current setting.
- Functions such as `gr()`, which need a ordering type as an argument, call `dp_ord()` internally during the execution. The setting remains after the execution.

    Fundamental arithmetics for distributed polynomial also use the current setting. Therefore, when such arithmetics for distributed polynomials are done, the current setting

must coincide with the ordering type which was used upon the creation of the poly-
nomials. It is assumed that such polynomials were generated under the same ordering
type.

- Type of term ordering must be correctly set by this function when functions other than
  top level functions are called directly.

- If the argument is a list, then an ordering type in a free module is set. If the argument
  is [0,Ord] then a TOP ordering based on the ordering type specified by Ord is set. If
  the argument is [1,Ord] then a POT ordering is set.

```
[19] dp_ord(0)$
[20] <<1,2,3>>+<<3,1,1>>;
(1)*<<1,2,3>>+(1)*<<3,1,1>>
[21] dp_ord(2)$
[22] <<1,2,3>>+<<3,1,1>>;
(1)*<<3,1,1>>+(1)*<<1,2,3>>
```

References
        Section 8.4 [Setting term orderings], page 126

8.10.12 dp_set_weight, dp_set_top_weight, dp_weyl_set_weight

dp_set_weight([*weight*])
        :: Set and show the sugar weight.

dp_set_top_weight([*weight*])
        :: Set and show the top weight.

dp_weyl_set_weight([*weight*])
        :: Set and show the weyl weight.

return   a vector

weight   a list or vector of integers

- dp_set_weight sets the sugar weight=*weight*. It returns the current sugar weight. A
  sugar weight is a vector with positive integer components and it represents the weights
  of variables. It is used for computing the weight of a monomial in a graded ordering.
  It is recommended to append a component 1 at the end of the weight vector for a
  homogenizing variable.

- dp_set_top_weight sets the top weight=*weight*. It returns the current top weight. It
  a top weight is set, the weights of monomials under the top weight are firstly compared.
  If the the weights are equal then the current term ordering is applied as a tie breaker,
  but the top weight is not used in the tie breaker.

- dp_weyl_set_weight sets the weyl weigh=*weight*. It returns the current weyl weight.
  If a weyl weight w is set, in the comparison by the term order type 11, a term order
  with the top weight=(-w,w) and the tie breaker=graded reverse lex is applied.

References
        Section 8.5 [Weight], page 128

8.10.13 `dp_ptod`

`dp_ptod(`*poly*`,`*vlist*`)`
                :: Converts an ordinary polynomial into a distributed polynomial.

return   distributed polynomial

poly   polynomial

vlist   list

- According to the variable ordering *vlist* and current type of term ordering, this function converts an ordinary polynomial into a distributed polynomial.
- Indeterminates not included in *vlist* are regarded to belong to the coefficient field.

```
[50] dp_ord(0);
1
[51] dp_ptod((x+y+z)^2,[x,y,z]);
(1)*<<2,0,0>>+(2)*<<1,1,0>>+(1)*<<0,2,0>>+(2)*<<1,0,1>>+(2)*<<0,1,1>>
+(1)*<<0,0,2>>
[52] dp_ptod((x+y+z)^2,[x,y]);
(1)*<<2,0>>+(2)*<<1,1>>+(1)*<<0,2>>+(2*z)*<<1,0>>+(2*z)*<<0,1>>
+(z^2)*<<0,0>>
```

References
                Section 8.10.17 [`dp_dtop`], page 143, Section 8.10.11 [`dp_ord`], page 140.

8.10.14 `dpm_dptodpm`

`dpm_dptodpm(`*dpoly*`,`*pos*`)`
                :: Converts a distributed polynomial into a module polynomial.

return   module polynomial

dpoly   distributed polynomial

pos   positive integer

- This function converts a distributed polynomial into a module polynomial.
- The output is `dpoly e_pos`.

```
[50] dp_ord([0,0])$
[51] D=dp_ptod((x+y+z)^2,[x,y,z]);
(1)*<<2,0,0>>+(2)*<<1,1,0>>+(1)*<<0,2,0>>+(2)*<<1,0,1>>+(2)*<<0,1,1>>
+(1)*<<0,0,2>>
[52] dp_dptodpm(D,2);
(1)*<<2,0,0:2>>+(2)*<<1,1,0:2>>+(1)*<<0,2,0:2>>+(2)*<<1,0,1:2>>
+(2)*<<0,1,1:2>>+(1)*<<0,0,2:2>>
```

References
                Section 8.10.13 [`dp_ptod`], page 142, Section 8.10.11 [`dp_ord`], page 140.

8.10.15 `dpm_ltod`

`dpm_ltod(`*plist*`,`*vlist*`)`
                :: Converts a list of polynomials into a module polynomial.

return   module polynomial

plist   list of polynomials

vlist   list of variables

- This function converts a list of polynomials into a module polynomial.
- `[p1,...,pm]` is converted into `p1 e1+...+pm em`.

```
[2126] dp_ord([0,0])$
[2127] dpm_ltod([x^2+y^2,x,y-z],[x,y,z]);
(1)*<<2,0,0:1>>+(1)*<<0,2,0:1>>+(1)*<<1,0,0:2>>+(1)*<<0,1,0:3>>
+(-1)*<<0,0,1:3>>
```

References
        Section 8.10.16 [`dpm_dtol`], page 143, Section 8.10.11 [`dp_ord`], page 140.

8.10.16 `dpm_dtol`

`dpm_dtol(`*poly*`,`*vlist*`)`
        :: Converts a module polynomial into a list of polynomials.

return   list of polynomials

poly   module polynomial

vlist   list of variables

- This function converts a module polynomial into a list of polynomials.
- `p1 e1+...+pm em` is converted into `[p1,...,pm]`.
- The length of the output list is equal to the largest index among those of the standard bases containd in `poly`.

```
[2126] dp_ord([0,0])$
[2127] D=(1)*<<2,0,0:1>>+(1)*<<0,2,0:1>>+(1)*<<1,0,0:2>>+(1)*<<0,1,0:3>>
+(-1)*<<0,0,1:3>>$
[2128] dpm_dtol(D,[x,y,z]);
[x^2+y^2,x,y-z]
```

References
        Section 8.10.15 [`dpm_ltod`], page 143, Section 8.10.11 [`dp_ord`], page 140.

8.10.17 `dp_dtop`

`dp_dtop(`*dpoly*`,`*vlist*`)`
        :: Converts a distributed polynomial into an ordinary polynomial.

return   polynomial

dpoly   distributed polynomial

vlist   list

- This function converts a distributed polynomial into an ordinary polynomial according to a list of indeterminates *vlist*.
- *vlist* is such a list that its length coincides with the number of variables of *dpoly*.

```
[53] T=dp_ptod((x+y+z)^2,[x,y]);
(1)*<<2,0>>+(2)*<<1,1>>+(1)*<<0,2>>+(2*z)*<<1,0>>+(2*z)*<<0,1>>
+(z^2)*<<0,0>>
[54] P=dp_dtop(T,[a,b]);
z^2+(2*a+2*b)*z+a^2+2*b*a+b^2
```

8.10.18 `dp_mod, dp_rat`

`dp_mod(`*p*`,`*mod*`,`*subst*`)`

:: Converts a distributed polynomial into one with coefficients in a finite field.

`dp_rat(`*p*`)`

:: Converts a distributed polynomial with coefficients in a finite field into one with coefficients in the rationals.

return	distributed polynomial
p	distributed polynomial
mod	prime
subst	list

- `dp_nf_mod()` and `dp_true_nf_mod()` require distributed polynomials with coefficients in a finite field as arguments. `dp_mod()` is used to convert distributed polynomials with rational number coefficients into appropriate ones. Polynomials with coefficients in a finite field cannot be used as inputs of operations with polynomials with rational number coefficients. `dp_rat()` is used for such cases.

- The ground finite field must be set in advance by using `setmod()`.

- *subst* is such a list as `[[`*var*`,`*value*`],...]`. This is valid when the ground field of the input polynomial is a rational function field. *var*'s are variables in the ground field and the list means that *value* is substituted for *var* before converting the coefficients into elements of a finite field.

`References`

Section 8.10.21 [`dp_nf dp_nf_mod dp_true_nf dp_true_nf_mod dp_weyl_nf dp_weyl_nf_mod`], page 146, Section 6.3.11 [`subst psubst`], page 52, Section 6.1.17 [`setmod`], page 44.

8.10.19 `dp_homo, dp_dehomo`

`dp_homo(`*dpoly*`)`

:: Homogenize a distributed polynomial

`dp_dehomo(`*dpoly*`)`

:: Dehomogenize a homogeneous distributed polynomial

return	distributed polynomial
dpoly	distributed polynomial

- `dp_homo()` makes a copy of *dpoly*, extends the length of the exponent vector of each term t in the copy by 1, and sets the value of the newly appended component to d-`deg(`*t*`)`, where d is the total degree of *dpoly*.

- `dp_dehomo()` make a copy of *dpoly* and removes the last component of each terms in the copy.

- Appropriate term orderings must be set when the results are used as inputs of some operations.

- These are used internally in `hgr()` etc.

  ```
  [202] X=<<1,2,3>>+3*<<1,2,1>>;
  (1)*<<1,2,3>>+(3)*<<1,2,1>>
  [203] dp_homo(X);
  (1)*<<1,2,3,0>>+(3)*<<1,2,1,2>>
  [204] dp_dehomo(@);
  (1)*<<1,2,3>>+(3)*<<1,2,1>>
  ```

References
        Section 8.10.1 [gr hgr gr_mod], page 131.

8.10.20 dp_ptozp, dp_prim

`dp_ptozp(dpoly)`
        :: Converts a distributed polynomial *poly* with rational coefficients into an integral distributed polynomial such that GCD of all its coefficients is 1.

`dp_prim(dpoly)`
        :: Converts a distributed polynomial *poly* with rational function coefficients into an integral distributed polynomial such that polynomial GCD of all its coefficients is 1.

return   distributed polynomial

dpoly   distributed polynomial

- `dp_ptozp()` executes the same operation as `ptozp()` for a distributed polynomial. If the coefficients include polynomials, polynomial contents included in the coefficients are not removed.

- `dp_prim()` removes polynomial contents.

  ```
  [208] X=dp_ptod(3*(x-y)*(y-z)*(z-x),[x]);
  (-3*y+3*z)*<<2>>+(3*y^2-3*z^2)*<<1>>+(-3*z*y^2+3*z^2*y)*<<0>>
  [209] dp_ptozp(X);
  (-y+z)*<<2>>+(y^2-z^2)*<<1>>+(-z*y^2+z^2*y)*<<0>>
  [210] dp_prim(X);
  (1)*<<2>>+(-y-z)*<<1>>+(z*y)*<<0>>
  ```

References
        Section 6.3.18 [ptozp], page 57.

8.10.21 `dp_nf`, `dp_nf_mod`, `dp_true_nf`, `dp_true_nf_mod`, `dp_weyl_nf`, `dp_weyl_nf_mod`

`dp_nf(`*indexlist*`,`*dpoly*`,`*dpolyarray*`,`*fullreduce*`)`
`dp_weyl_nf(`*indexlist*`,`*dpoly*`,`*dpolyarray*`,`*fullreduce*`)`
`dp_nf_mod(`*indexlist*`,`*dpoly*`,`*dpolyarray*`,`*fullreduce*`,`*mod*`)`
`dp_weyl_nf_mod(`*indexlist*`,`*dpoly*`,`*dpolyarray*`,`*fullreduce*`,`*mod*`)`
:: Computes the normal form of a distributed polynomial. (The result may be multiplied by a constant in the ground field.)

`dp_true_nf(`*indexlist*`,`*dpoly*`,`*dpolyarray*`,`*fullreduce*`)`
`dp_true_nf_mod(`*indexlist*`,`*dpoly*`,`*dpolyarray*`,`*fullreduce*`,`*mod*`)`
:: Computes the normal form of a distributed polynomial. (The true result is returned in such a list as `[numerator, denominator]`)

return   `dp_nf()` : distributed polynomial, `dp_true_nf()` : list

indexlist   list

dpoly   distributed polynomial

dpolyarray
          array of distributed polynomial

fullreduce   flag

mod   prime

- Computes the normal form of a distributed polynomial.

- Functions whose name contain `weyl` compute normal forms in Weyl algebra. The description below also applies to the functions for Weyl algebra.

- `dp_nf_mod()` and `dp_true_nf_mod()` require distributed polynomials with coefficients in a finite field as arguments.

- The result of `dp_nf()` may be multiplied by a constant in the ground field in order to make the result integral. The same is true for `dp_nf_mod()`, but it returns the true normal form if the ground field is a finite field.

- `dp_true_nf()` and `dp_true_nf_mod()` return such a list as `[`*nm*`,`*dn*`]`. Here *nm* is a distributed polynomial whose coefficients are integral in the ground field, *dn* is an integral element in the ground field and *nm*/*dn* is the true normal form.

- *dpolyarray* is a vector whose components are distributed polynomials and *indexlist* is a list of indices which is used for the normal form computation.

- When argument *fullreduce* has non-zero value, all terms are reduced. When it has value 0, only the head term is reduced.

- As for the polynomials specified by *indexlist*, one specified by an index placed at the preceding position has priority to be selected.

- In general, the result of the function may be different depending on *indexlist*. However, the result is unique for Groebner bases.

- These functions are useful when a fixed non-distributed polynomial set is used as a set of reducers to compute normal forms of many polynomials. For single computation `p_nf` and `p_true_nf` are sufficient.

```
[0] load("gr")$
[64] load("katsura")$
[69] K=katsura(4)$
[70] dp_ord(2)$
[71] V=[u0,u1,u2,u3,u4]$
[72] DP1=newvect(length(K),map(dp_ptod,K,V))$
[73] G=gr(K,V,2)$
[74] DP2=newvect(length(G),map(dp_ptod,G,V))$
[75] T=dp_ptod((u0-u1+u2-u3+u4)^2,V)$
[76] dp_dtop(dp_nf([0,1,2,3,4],T,DP1,1),V);
u4^2+(6*u3+2*u2+6*u1-2)*u4+9*u3^2+(6*u2+18*u1-6)*u3+u2^2
+(6*u1-2)*u2+9*u1^2-6*u1+1
[77] dp_dtop(dp_nf([4,3,2,1,0],T,DP1,1),V);
-5*u4^2+(-4*u3-4*u2-4*u1)*u4-u3^2-3*u3-u2^2+(2*u1-1)*u2-2*u1^2-3*u1+1
[78] dp_dtop(dp_nf([0,1,2,3,4],T,DP2,1),V);
-1138087976845165778088612297273078520347097001020471455633353049221 0
4567759300057165055600620871509284008761502170798203114394775605875 83
488*u4^15+...
[79] dp_dtop(dp_nf([4,3,2,1,0],T,DP2,1),V);
-1138087976845165778088612297273078520347097001020471455633353049221 0
4567759300057165055600620871509284008761502170798203114394775605875 83
488*u4^15+...
[80] @78==@79;
1
```

References

Section 8.10.17 [dp_dtop], page 143, Section 8.10.11 [dp_ord], page 140, Section 8.10.18 [dp_mod dp_rat], page 144, Section 8.10.40 [p_nf p_nf_mod p_true_nf p_true_nf_mod], page 156.

8.10.22 `dpm_nf`, `dpm_nf_and_quotient`

`dpm_nf([indexlist,]dpoly,dpolyarray,fullreduce)`
          :: Computes the normal form of a module polynomial. (The result may be multiplied by a constant in the ground field.)

`dpm_nf_and_quotient([indexlist,]dpoly,dpolyarray)`
          :: Computes the normal form of a module polynomial and the quotient.

return   `dpm_nf()` : module polynomial, `dpm_nf_and_quotient()` : list

indexlist   list

dpoly   module polynomial

dpolyarray
          array of module polynomial

- Computes the normal form of a module polynomial.
- The result of `dpm_nf()` may be multiplied by a constant in the ground field in order to make the result integral.

- *dpolyarray* is a vector whose components are module polynomials and *indexlist* is a list of indices which is used for the normal form computation.

- If *indexlist* is given, only the polynomials in *dpolyarray* specified in *indexlist* is used in the division. An index placed at the preceding position has priority to be selected. If *indexlist* is not given, all the polynomials in *dpolyarray* are used.

- `dpm_nf_and_quotient()` returns such a list as `[nm,dn,quo]`. Here *nm* is a module polynomial whose coefficients are integral in the ground field, *dn* is an integral element in the ground field and *nm/dn* is the true normal form. *quo* is an array containing the quotients of the division satisfying *dndpoly=nm+quo[0]dpolyarray[0]+....*

- When argument *fullreduce* has non-zero value, all terms are reduced. When it has value 0, only the head term is reduced.

```
[2126] dp_ord([1,0])$
[2127] S=ltov([(1)*<<0,0,2,0:1>>+(1)*<<0,0,1,1:1>>+(1)*<<0,0,0,2:1>>
+(-1)*<<3,0,0,0:2>>+(-1)*<<0,0,2,1:2>>+(-1)*<<0,0,1,2:2>>
+(1)*<<3,0,1,0:3>>+(1)*<<3,0,0,1:3>>+(1)*<<0,0,2,2:3>>,
(-1)*<<0,1,0,0:1>>+(-1)*<<0,0,1,0:1>>+(-1)*<<0,0,0,1:1>>
+(-1)*<<3,0,0,0:3>>+(1)*<<0,1,1,1:3>>,(1)*<<0,1,0,0:2>>
+(1)*<<0,0,1,0:2>>+(1)*<<0,0,0,1:2>>+(-1)*<<0,1,1,0:3>>
+(-1)*<<0,1,0,1:3>>+(-1)*<<0,0,1,1:3>>])$
[2128] U=dpm_sp(S[0],S[1]);
(1)*<<0,0,3,0:1>>+(-1)*<<0,1,1,1:1>>+(1)*<<0,0,2,1:1>>
+(-1)*<<0,1,0,2:1>>+(1)*<<3,1,0,0:2>>+(1)*<<0,1,2,1:2>>
+(1)*<<0,1,1,2:2>>+(-1)*<<3,1,1,0:3>>+(1)*<<3,0,2,0:3>>
+(-1)*<<3,1,0,1:3>>+(-1)*<<0,1,3,1:3>>+(-1)*<<0,1,2,2:3>>
[2129] dpm_nf(U,S,1);
0
[2130] L=dpm_nf_and_quotient(U,S)$
[2131] Q=L[2]$
[2132] D=L[1]$
[2133] D*U-(Q[1]*S[1]+Q[2]*S[2]);
0
```

References
         ⟨undefined⟩ [dpm_sp], page ⟨undefined⟩, Section 8.10.11 [dp_ord], page 140.

8.10.23 dp_hm, dp_ht, dp_hc, dp_rest

dp_hm(*dpoly*)
         :: Gets the head monomial.

dp_ht(*dpoly*)
         :: Gets the head term.

dp_hc(*dpoly*)
         :: Gets the head coefficient.

dp_rest(*dpoly*)
         :: Gets the remainder of the polynomial where the head monomial is removed.

return   `dp_hm()`, `dp_ht()`, `dp_rest()` : distributed polynomial `dp_hc()` : number or polynomial

dpoly   distributed polynomial

- These are used to get various parts of a distributed polynomial.
- The next equations hold for a distributed polynomial *p*.

    `p = dp_hm(p) + dp_rest(p)`
    `dp_hm(p) = dp_hc(p) dp_ht(p)`
    ```
    [87] dp_ord(0)$
    [88] X=ptozp((a46^2+7/10*a46+7/48)*u3^4-50/27*a46^2-35/27*a46-49/216)$
    [89] T=dp_ptod(X,[u3,u4,a46])$
    [90] dp_hm(T);
    (2160)*<<4,0,2>>
    [91] dp_ht(T);
    (1)*<<4,0,2>>
    [92] dp_hc(T);
    2160
    [93] dp_rest(T);
    (1512)*<<4,0,1>>+(315)*<<4,0,0>>+(-4000)*<<0,0,2>>+(-2800)*<<0,0,1>>
    +(-490)*<<0,0,0>>
    ```

8.10.24 `dpm_hm, dpm_ht, dpm_hc, dpm_hp, dpm_rest`

`dpm_hm(dpoly)`
        :: Gets the head monomial of a module polynomial.

`dpm_ht(dpoly)`
        :: Gets the head term of a module polynomial.

`dpm_hc(dpoly)`
        :: Gets the head coefficient of a module polynomial.

`dpm_hp(dpoly)`
        :: Gets the head position of a module polynomial.

`dpm_rest(dpoly)`
        :: Gets the remainder of a module polynomial where the head monomial is removed.

return   `dpm_hm()`, `dpm_ht()`, `dpm_rest()` : module polynomial `dpm_hc()` : monomial

dpoly   distributed polynomial

- These are used to get various parts of a module polynomial.
- `dpm_hc()` returns the monomial that is the coefficient of `dpm_hm()` with respect to the standard base. For getting its scalar coefficient apply `dp_hc()`.
- `dpm_hp()` returns the index of the standard base conteind in the head module monomial.

    ```
    [2126] dp_ord([1,0]);
    [1,0]
    [2127] F=2*<<1,2,0:2>>-3*<<1,0,2:3>>+<<2,1,0:2>>;
    (1)*<<2,1,0:2>>+(2)*<<1,2,0:2>>+(-3)*<<1,0,2:3>>
    ```

```
[2128] M=dpm_hm(F);
(1)*<<2,1,0:2>>
[2129] C=dpm_hc(F);
(1)*<<2,1,0>>
[2130] R=dpm_rest(F);
(2)*<<1,2,0:2>>+(-3)*<<1,0,2:3>>
[2131] dpm_hp(F);
2
```

8.10.25 `dp_td`, `dp_sugar`

`dp_td(`*dpoly*`)`
            :: Gets the total degree of the head term.

`dp_sugar(`*dpoly*`)`
            :: Gets the `sugar` of a polynomial.

return   non-negative integer

dpoly   distributed polynomial

onoff   flag

- Function `dp_td()` returns the total degree of the head term, i.e., the sum of all exponent of variables in that term.
- Upon creation of a distributed polynomial, an integer called `sugar` is associated. This value is the total degree of the virtually homogenized one of the original polynomial.
- The quantity `sugar` is an important guide to determine the selection strategy of critical pairs in Groebner basis computation.

```
[74] dp_ord(0)$
[75] X=<<1,2>>+<<0,1>>$
[76] Y=<<1,2>>+<<1,0>>$
[77] Z=X-Y;
(-1)*<<1,0>>+(1)*<<0,1>>
[78] dp_sugar(T);
3
```

8.10.26 `dp_lcm`

`dp_lcm(`*dpoly1*`,`*dpoly2*`)`
            :: Returns the least common multiple of the head terms of the given two polynomials.

return   distributed polynomial

dpoly1 dpoly2
            distributed polynomial

- Returns the least common multiple of the head terms of the given two polynomials, where coefficient is always set to 1.

```
[100] dp_lcm(<<1,2,3,4,5>>,<<5,4,3,2,1>>);
```

```
(1)*<<5,4,3,4,5>>
```

References
>   Section 8.10.40 [p_nf p_nf_mod p_true_nf p_true_nf_mod], page 156.

8.10.27 dp_redble

`dp_redble(dpoly1,dpoly2)`
>   :: Checks whether one head term is divisible by the other head term.

return   integer

dpoly1 dpoly2
>   distributed polynomial

- Returns 1 if the head term of *dpoly2* divides the head term of *dpoly1*; otherwise 0.
- Used for finding candidate terms at reduction of polynomials.

```
[148] C;
(1)*<<1,1,1,0,0>>+(1)*<<0,1,1,1,0>>+(1)*<<1,1,0,0,1>>+(1)*<<1,0,0,1,1>>
[149] T;
(3)*<<2,1,0,0,0>>+(3)*<<1,2,0,0,0>>+(1)*<<0,3,0,0,0>>+(6)*<<1,1,1,0,0>>
[150] for ( ; T; T = dp_rest(T)) print(dp_redble(T,C));
0
0
0
1
```

References
>   Section 8.10.33 [dp_red dp_red_mod], page 153.

8.10.28 dpm_redble

`dpm_redble(dpoly1,dpoly2)`
>   :: Checks whether one head term is divisible by the other head term.

return   integer

dpoly1 dpoly2
>   module polynomial

- Returns 1 if the head term of *dpoly2* divides the head term of *dpoly1*; otherwise 0.
- Used for finding candidate terms at reduction of polynomials.

8.10.29 dp_subd

`dp_subd(dpoly1,dpoly2)`
>   :: Returns the quotient monomial of the head terms.

return   distributed polynomial

dpoly1 dpoly2
>   distributed polynomial

- Gets `dp_ht(dpoly1)/dp_ht(dpoly2)`. The coefficient of the result is always set to 1.
- Divisibility assumed.

```
[162] dp_subd(<<1,2,3,4,5>>,<<1,1,2,3,4>>);
(1)*<<0,1,1,1,1>>
```

References
        Section 8.10.33 [dp_red dp_red_mod], page 153.

8.10.30 dp_vtoe, dp_etov

dp_vtoe(*vect*)
        :: Converts an exponent vector into a term.

dp_etov(*dpoly*)
        :: Convert the head term of a distributed polynomial into an exponent vector.

return   dp_vtoe : distributed polynomial, dp_etov : vector

vect   vector

dpoly   distributed polynomial

- dp_vtoe() generates a term whose exponent vector is *vect*.
- dp_etov() generates a vector which is the exponent vector of the head term of dpoly.

```
[211] X=<<1,2,3>>;
(1)*<<1,2,3>>
[212] V=dp_etov(X);
[ 1 2 3 ]
[213] V[2]++$
[214] Y=dp_vtoe(V);
(1)*<<1,2,4>>
```

8.10.31 dp_mbase

dp_mbase(*dplist*)
        :: Computes the monomial basis

return   list of distributed polynomial

dplist   list of distributed polynomial

- Assuming that *dplist* is a list of distributed polynomials which is a Groebner basis with respect to the current ordering type and that the ideal *I* generated by *dplist* in K[X] is zero-dimensional, this function computes the monomial basis of a finite dimensional K-vector space K[X]/I.
- The number of elements in the monomial basis is equal to the K-dimenstion of K[X]/I.

```
[215] K=katsura(5)$
[216] V=[u5,u4,u3,u2,u1,u0]$
[217] G0=gr(K,V,0)$
[218] H=map(dp_ptod,G0,V)$
[219] map(dp_ptod,dp_mbase(H),V)$
[u0^5,u4*u0^3,u3*u0^3,u2*u0^3,u1*u0^3,u0^4,u3^2*u0,u2*u3*u0,u1*u3*u0,
u1*u2*u0,u1^2*u0,u4*u0^2,u3*u0^2,u2*u0^2,u1*u0^2,u0^3,u3^2,u2*u3,u1*u3,
u1*u2,u1^2,u4*u0,u3*u0,u2*u0,u1*u0,u0^2,u4,u3,u2,u1,u0,1]
```

References
        Section 8.10.1 [gr hgr gr_mod], page 131.

8.10.32 `dp_mag`

`dp_mag(`*p*`)`

:: Computes the sum of bit lengths of coefficients of a distributed polynomial.

return   integer

p   distributed polynomial

- This function computes the sum of bit lengths of coefficients of a distributed polynomial *p*. If a coefficient is non integral, the sum of bit lengths of the numerator and the denominator is taken.
- This is a measure of the size of a polynomial. Especially for zero-dimensional system coefficient swells are often serious and the returned value is useful to detect such swells.
- If `ShowMag` and `Print` for `dp_gr_flags()` are on, values of `dp_mag()` for intermediate basis elements are shown.

```
[221] X=dp_ptod((x+2*y)^10,[x,y])$
[222] dp_mag(X);
115
```

`References`

Section 8.10.10 [`dp_gr_flags dp_gr_print`], page 140.

8.10.33 `dp_red, dp_red_mod`

`dp_red(`*dpoly1*`,`*dpoly2*`,`*dpoly3*`)`
`dp_red_mod(`*dpoly1*`,`*dpoly2*`,`*dpoly3*`,`*mod*`)`
:: Single reduction operation

return   list

dpoly1 dpoly2 dpoly3
                distributed polynomial

vlist   list

mod   prime

- Reduces a distributed polynomial, *dpoly1* + *dpoly2*, by *dpoly3* for single time.
- An input for `dp_red_mod()` must be converted into a distributed polynomial with coefficients in a finite field.
- This implies that the divisibility of the head term of *dpoly2* by the head term of *dpoly3* is assumed.
- When integral coefficients, computation is so carefully performed that no rational operations appear in the reduction procedure. It is computed for integers *a* and *b*, and a term *t* as: $a(dpoly1 + dpoly2)-bt\ dpoly3$.
- The result is a list `[a `*dpoly1*`,a `*dpoly2*` - bt `*dpoly3*`]`.

```
[157] D=(3)*<<2,1,0,0,0>>+(3)*<<1,2,0,0,0>>+(1)*<<0,3,0,0,0>>;
(3)*<<2,1,0,0,0>>+(3)*<<1,2,0,0,0>>+(1)*<<0,3,0,0,0>>
[158] R=(6)*<<1,1,1,0,0>>;
(6)*<<1,1,1,0,0>>
[159] C=12*<<1,1,1,0,0>>+(1)*<<0,1,1,1,0>>+(1)*<<1,1,0,0,1>>;
```

```
    (12)*<<1,1,1,0,0>>+(1)*<<0,1,1,1,0>>+(1)*<<1,1,0,0,1>>
[160] dp_red(D,R,C);
[(6)*<<2,1,0,0,0>>+(6)*<<1,2,0,0,0>>+(2)*<<0,3,0,0,0>>,
(-1)*<<0,1,1,1,0>>+(-1)*<<1,1,0,0,1>>]
```

References
          Section 8.10.18 [dp_mod dp_rat], page 144.

8.10.34 dp_sp, dp_sp_mod

dp_sp(*dpoly1*,*dpoly2*)
dp_sp_mod(*dpoly1*,*dpoly2*,*mod*)
          :: Computation of an S-polynomial

return   distributed polynomial

dpoly1 dpoly2
          distributed polynomial

mod   prime
  - This function computes the S-polynomial of *dpoly1* and *dpoly2*.
  - Inputs of `dp_sp_mod()` must be polynomials with coefficients in a finite field.
  - The result may be multiplied by a constant in the ground field in order to make the
    result integral.
    ```
    [227] X=dp_ptod(x^2*y+x*y,[x,y]);
    (1)*<<2,1>>+(1)*<<1,1>>
    [228] Y=dp_ptod(x*y^2+x*y,[x,y]);
    (1)*<<1,2>>+(1)*<<1,1>>
    [229] dp_sp(X,Y);
    (-1)*<<2,1>>+(1)*<<1,2>>
    ```

References
          Section 8.10.18 [dp_mod dp_rat], page 144.

8.10.35 dpm_sp

dpm_sp(*dpoly1*,*dpoly2*[|coef=1])
          :: Computation of an S-polynomial

return   module polynomial or list

dpoly1 dpoly2
          module polynomial
  - This function computes the S-polynomial of *dpoly1* and *dpoly2*.
  - If an option *coef=1* is specified, it returns a list [S,t1,t2], where S is the S-polynmial
    and t1, t2 are monomials satisfying S=t1 dpoly1-t2 dpoly2.

8.10.36 dpm_schreyer_base

dpm_schreyer_base(*G*)
          :: Computation of a Groebner basis of the syzygy module

return   list of module polynomials

G   list of module polynomials

- This function computes a minimal Groebner basis of *syz(G)* for a Groenber basis *G* that is represented as a list of module polynomials.
- The result is a Groebner basis with respect to a Schreyer ordering determined by the leading terms of *G*. As a side effect, this Schreyer ordering is autoatically set.

8.10.37 `dpm_schreyer_frame`

`dpm_schreyer_frame(G)`
            :: Computation of the Schreyer frame

return   a list of lists of module monomials

G   list of module polynomials

- This function computes the Schreyer frame starting from a Groebner basis *G*, that is the lists of leading monomials of Groebner bases of syzygy modules with respect to Schreyer orderings in the Schreyer free resolution.
- The result is a list $[M\_m,...,M\_1]$, where M_i is the list of leading monomials of the images of standard bases of the free module F_i in the Schreyer free resolution.
- As a by-product, data for setting a Schreyer order in each level are created. The date are used by `dpm_set_schreyer_level` for setting a Schreyer order in each level.

8.10.38 `dpm_set_schreyer_level`

`dpm_set_schreyer_level(L)`
            :: Setting the Schreyer ordering of a specified level

return   a non-negative integer

G   a non-negative integer

- This function computes the Schreyer frame starting from a Groebner basis *G*, that is the lists of leading monomials of Groebner bases of syzygy modules with respect to Schreyer orderings in the Schreyer free resolution.
- The result is a list $[M\_m,...,M\_1]$, where M_i is the list of leading monomials of the images of standard bases of the free module F_i in the Schreyer free resolution.
- As a by-product, data for setting a Schreyer order in each level are created. The date are used by `dpm_set_schreyer_level` for setting a Schreyer order in each level.

8.10.39 `dpm_sp_nf`

`dpm_sp_nf(C,Z,P,Q)`
            :: Computation of a remainder of an S-polynomial modulo a polynomial array

return   a list of module monomials

C   an array of module polynomials

Z   an array of integer lists

P

Q   integers

- When the remainder of the S-polynomial of *C[P]* and *C[Q]* modulo *C* is represented as

$$ctC[P] - c't'C[Q] = g_1C[1] + \cdots + g_LC[L] + f$$

  this function returns a list [g',f], where

$$g' = cte_P - c't'e_Q - (g_1e_1 + ... + g_Le_L).$$

- The *I*-th element of an array *Z* is a list of indices of elements of *C* whose leading position is *I*.

8.10.40 `p_nf, p_nf_mod, p_true_nf, p_true_nf_mod`

`p_nf(`*poly*`,`*plist*`,`*vlist*`,`*order*`)`
`p_nf_mod(`*poly*`,`*plist*`,`*vlist*`,`*order*`,`*mod*`)`
          :: Computes the normal form of the given polynomial. (The result may be multiplied by a constant.)

`p_true_nf(`*poly*`,`*plist*`,`*vlist*`,`*order*`)`
`p_true_nf_mod(`*poly*`,`*plist*`,`*vlist*`,`*order*`,`*mod*`)`
          :: Computes the normal form of the given polynomial. (The result is returned as a form of [`numerator, denominator`])

return   `p_nf` : polynomial, `p_true_nf` : list

poly   polynomial

plist vlist   list

order   number, list or matrix

mod   prime
  - Defined in the package '`gr`'.
  - Obtains the normal form of a polynomial by a polynomial list.
  - These are interfaces to `dp_nf()`, `dp_true_nf()`, `dp_nf_mod()`, `dp_true_nf_mod`
  - The polynomial *poly* and the polynomials in *plist* is converted, according to the variable ordering *vlist* and type of term ordering *otype*, into their distributed polynomial counterparts and passed to `dp_nf()`.
  - `dp_nf()`, `dp_true_nf()`, `dp_nf_mod()` and `dp_true_nf_mod()` is called with value 1 for *fullreduce*.
  - The result is converted back into an ordinary polynomial.
  - As for `p_true_nf()`, `p_true_nf_mod()` refer to `dp_true_nf()` and `dp_true_nf_mod()`.
    ```
    [79] K = katsura(5)$
    [80] V = [u5,u4,u3,u2,u1,u0]$
    [81] G = hgr(K,V,2)$
    [82] p_nf(K[1],G,V,2);
    0
    [83] L = p_true_nf(K[1]+1,G,V,2);
    [-1503...,-1503...]
    [84] L[0]/L[1];
    ```

```
     1
```

References

Section 8.10.13 [dp_ptod], page 142, Section 8.10.17 [dp_dtop], page 143, Section 8.10.11 [dp_ord], page 140, Section 8.10.21 [dp_nf dp_nf_mod dp_true_nf dp_true_nf_mod dp_weyl_nf dp_weyl_nf_mod], page 146.

8.10.41 p_terms

p_terms(*poly*,*vlist*,*order*)
          :: Monomials appearing in the given polynomial is collected into a list.

return   list

poly   polynomial

vlist   list

order   number, list or matrix

- Defined in the package 'gr'.

- This returns a list which contains all non-zero monomials in the given polynomial. The monomials are ordered according to the current type of term ordering and *vlist*.

- Since polynomials in a Groebner base often have very large coefficients, examining a polynomial as it is may sometimes be difficult to perform. For such a case, this function enables to examine which term is really exists.

```
[233] G=gr(katsura(5),[u5,u4,u3,u2,u1,u0],2)$
[234] p_terms(G[0],[u5,u4,u3,u2,u1,u0],2);
[u5,u0^31,u0^30,u0^29,u0^28,u0^27,u0^26,u0^25,u0^24,u0^23,u0^22,
u0^21,u0^20,u0^19,u0^18,u0^17,u0^16,u0^15,u0^14,u0^13,u0^12,u0^11,
u0^10,u0^9,u0^8,u0^7,u0^6,u0^5,u0^4,u0^3,u0^2,u0,1]
```

8.10.42 gb_comp

gb_comp(*plist1*, *plist2*)
          :: Checks whether two polynomial lists are equal or not as a set

return 0 or 1
plist1 plist2

- This function checks whether *plist1* and *plist2* are equal or not as a set .

- For the same input and the same term ordering different functions for Groebner basis computations may produce different outputs as lists. This function compares such lists whether they are equal as a generating set of an ideal.

```
[243] C=cyclic(6)$
[244] V=[c0,c1,c2,c3,c4,c5]$
[245] G0=gr(C,V,0)$
[246] G=tolex(G0,V,0,V)$
[247] GG=lex_tl(C,V,0,V,0)$
[248] gb_comp(G,GG);
     1
```

8.10.43 `katsura`, `hkatsura`, `cyclic`, `hcyclic`

`katsura(`*n*`)`
`hkatsura(`*n*`)`
`cyclic(`*n*`)`
`hcyclic(`*n*`)`
          :: Generates a polynomial list of standard benchmark.

return   list

n   integer

- Function `katsura()` is defined in '`katsura`', and function `cyclic()` in '`cyclic`'.

- These functions generate a series of polynomial sets, respectively, which are often used for testing and bench marking: `katsura`, `cyclic` and their homogenized versions.

- Polynomial set `cyclic` is sometimes called by other name: `Arnborg`, `Lazard`, and `Davenport`.

```
[74] load("katsura")$
[79] load("cyclic")$
[89] katsura(5);
[u0+2*u4+2*u3+2*u2+2*u1+2*u5-1,2*u4*u0-u4+2*u1*u3+u2^2+2*u5*u1,
2*u3*u0+2*u1*u4-u3+(2*u1+2*u5)*u2,2*u2*u0+2*u2*u4+(2*u1+2*u5)*u3
-u2+u1^2,2*u1*u0+(2*u3+2*u5)*u4+2*u2*u3+2*u1*u2-u1,
u0^2-u0+2*u4^2+2*u3^2+2*u2^2+2*u1^2+2*u5^2]
[90] hkatsura(5);
[-t+u0+2*u4+2*u3+2*u2+2*u1+2*u5,
-u4*t+2*u4*u0+2*u1*u3+u2^2+2*u5*u1,-u3*t+2*u3*u0+2*u1*u4+(2*u1+2*u5)*u2,
-u2*t+2*u2*u0+2*u2*u4+(2*u1+2*u5)*u3+u1^2,
-u1*t+2*u1*u0+(2*u3+2*u5)*u4+2*u2*u3+2*u1*u2,
-u0*t+u0^2+2*u4^2+2*u3^2+2*u2^2+2*u1^2+2*u5^2]
[91] cyclic(6);
[c5*c4*c3*c2*c1*c0-1,
((((c4+c5)*c3+c5*c4)*c2+c5*c4*c3)*c1+c5*c4*c3*c2)*c0+c5*c4*c3*c2*c1,
(((c3+c5)*c2+c5*c4)*c1+c5*c4*c3)*c0+c4*c3*c2*c1+c5*c4*c3*c2,
((c2+c5)*c1+c5*c4)*c0+c3*c2*c1+c4*c3*c2+c5*c4*c3,
(c1+c5)*c0+c2*c1+c3*c2+c4*c3+c5*c4,c0+c1+c2+c3+c4+c5]
[92] hcyclic(6);
[-c^6+c5*c4*c3*c2*c1*c0,
((((c4+c5)*c3+c5*c4)*c2+c5*c4*c3)*c1+c5*c4*c3*c2)*c0+c5*c4*c3*c2*c1,
(((c3+c5)*c2+c5*c4)*c1+c5*c4*c3)*c0+c4*c3*c2*c1+c5*c4*c3*c2,
((c2+c5)*c1+c5*c4)*c0+c3*c2*c1+c4*c3*c2+c5*c4*c3,
(c1+c5)*c0+c2*c1+c3*c2+c4*c3+c5*c4,c0+c1+c2+c3+c4+c5]
```

`References`
          Section 8.10.17 [`dp_dtop`], page 143.

8.10.44 primadec, primedec

primadec(*plist*,*vlist*)
primedec(*plist*,*vlist*)
            :: Computes decompositions of ideals.

return
plist   list of polynomials

vlist   list of variables

- A new package 'noro_pd.rr' provides more efficient functions for ideal decomposition.
- Function primadec() and primedec are defined in 'primdec'.
- primadec(), primedec() are the function for primary ideal decomposition and prime decomposition of the radical over the rationals respectively.
- The arguments are a list of polynomials and a list of variables. These functions accept ideals with rational function coefficients only.
- primadec returns the list of pair lists consisting a primary component and its associated prime.
- primedec returns the list of prime components.
- Each component is a Groebner basis and the corresponding term order is indicated by the global variables PRIMAORD, PRIMEORD respectively.
- primadec implements the primary decomposition algorithm in [Shimoyama,Yokoyama].
- If one only wants to know the prime components of an ideal, then use primedec because primadec may need additional costs if an input ideal is not radical.

```
[84] load("primdec")$
[102] primedec([p*q*x-q^2*y^2+q^2*y,-p^2*x^2+p^2*x+p*q*y,
(q^3*y^4-2*q^3*y^3+q^3*y^2)*x-q^3*y^4+q^3*y^3,
-q^3*y^4+2*q^3*y^3+(-q^3+p*q^2)*y^2],[p,q,x,y]);
[[y,x],[y,p],[x,q],[q,p],[x-1,q],[y-1,p],[(y-1)*x-y,q*y^2-2*q*y-p+q]]
[103] primadec([x,z*y,w*y^2,w^2*y-z^3,y^3],[x,y,z,w]);
[[[x,z*y,y^2,w^2*y-z^3],[z,y,x]],[[w,x,z*y,z^3,y^3],[w,z,y,x]]]
```

References
            Section 6.3.15 [fctr sqfr], page 54, Section 8.4 [Setting term orderings], page 126.

8.10.45 primedec_mod

primedec_mod(*plist*,*vlist*,*ord*,*mod*,*strategy*)
            :: Computes decompositions of ideals over small finite fields.

return
plist   list of polynomials

vlist   list of variables

ord   number, list or matrix

mod   positive integer

strategy   integer

- Function `primedec_mod()` is defined in '`primdec_mod`' and implements the prime decomposition algorithm in `[Yokoyama]`.

- `primedec_mod()` is the function for prime ideal decomposition of the radical of a polynomial ideal over small finite field, and they return a list of prime ideals, which are associated primes of the input ideal.

- `primedec_mod()` gives the decomposition over GF(*mod*). The generators of each resulting component consists of integral polynomials.

- Each resulting component is a Groebner basis with respect to a term order specified by [*vlist,ord*].

- If *strategy* is non zero, then the early termination strategy is tried by computing the intersection of obtained components incrementally. In general, this strategy is useful when the krull dimension of the ideal is high, but it may add some overhead if the dimension is small.

- If you want to see internal information during the computation, execute `dp_gr_print(2)` in advance.

  ```
  [0] load("primdec_mod")$
  [246] PP444=[x^8+x^2+t,y^8+y^2+t,z^8+z^2+t]$
  [247] primedec_mod(PP444,[x,y,z,t],0,2,1);
  [[y+z,x+z,z^8+z^2+t],[x+y,y^2+y+z^2+z+1,z^8+z^2+t],
  [y+z+1,x+z+1,z^8+z^2+t],[x+z,y^2+y+z^2+z+1,z^8+z^2+t],
  [y+z,x^2+x+z^2+z+1,z^8+z^2+t],[y+z+1,x^2+x+z^2+z+1,z^8+z^2+t],
  [x+z+1,y^2+y+z^2+z+1,z^8+z^2+t],[y+z+1,x+z,z^8+z^2+t],
  [x+y+1,y^2+y+z^2+z+1,z^8+z^2+t],[y+z,x+z+1,z^8+z^2+t]]
  [248]
  ```

`References`

Section 6.3.17 [`modfctr`], page 56, Section 8.10.6 [`dp_gr_main dp_gr_mod_main dp_gr_f_main dp_weyl_gr_main dp_weyl_gr_mod_main dp_weyl_gr_f_main`], page 136, Section 8.4 [`Setting term orderings`], page 126, Section 8.10.10 [`dp_gr_flags dp_gr_print`], page 140.

8.10.46 `bfunction, bfct, generic_bfct, ann, ann0`

`bfunction(`*f*`)`
`bfct(`*f*`)`
`generic_bfct(`*plist,vlist,dvlist,weight*`)`
          :: Computes the global *b* function of a polynomial or an ideal

`ann(`*f*`)`
`ann0(`*f*`)`   :: Computes the annihilator of a power of polynomial

return   polynomial or list

f   polynomial

plist   list of polynomials

vlist dvlist

          list of variables

- These functions are defined in 'bfct'.
- bfunction(*f*) and bfct(*f*) compute the global *b*-function b(s) of a polynomial *f*. b(s) is a polynomial of the minimal degree such that there exists P(x,s) in D[s], which is a polynomial ring over Weyl algebra D, and P(x,s)f^(s+1)=b(s)f^s holds.
- generic_bfct(*f*,*vlist*,*dvlist*,*weight*) computes the global *b*-function of a left ideal I in D generated by *plist*, with respect to *weight*. *vlist* is the list of x-variables, *vlist* is the list of corresponding D-variables.
- bfunction(*f*) and bfct(*f*) implement different algorithms and the efficiency depends on inputs.
- ann(*f*) returns the generator set of the annihilator ideal of *f*^s. ann(*f*) returns a list [*a*,*list*], where *a* is the minimal integral root of the global *b*-function of *f*, and *list* is a list of polynomials obtained by substituting s in ann(*f*) with *a*.
- See [Saito,Sturmfels,Takayama] for the details.

```
[0] load("bfct")$
[216] bfunction(x^3+y^3+z^3+x^2*y^2*z^2+x*y*z);
-9*s^5-63*s^4-173*s^3-233*s^2-154*s-40
[217] fctr(@);
[[-1,1],[s+2,1],[3*s+4,1],[3*s+5,1],[s+1,2]]
[218] F = [4*x^3*dt+y*z*dt+dx,x*z*dt+4*y^3*dt+dy,
x*y*dt+5*z^4*dt+dz,-x^4-z*y*x-y^4-z^5+t]$
[219] generic_bfct(F,[t,z,y,x],[dt,dz,dy,dx],[1,0,0,0]);
20000*s^10-70000*s^9+101750*s^8-79375*s^7+35768*s^6-9277*s^5
+1278*s^4-72*s^3
[220] P=x^3-y^2$
[221] ann(P);
[2*dy*x+3*dx*y^2,-3*dx*x-2*dy*y+6*s]
[222] ann0(P);
[-1,[2*dy*x+3*dx*y^2,-3*dx*x-2*dy*y-6]]
```

References

    Section 8.8 [Weyl algebra], page 130.

9 Algebraic numbers

9.1 Representation of algebraic numbers

In **Asir** algebraic number fields are not defined as independent objects. Instead, individual algebraic numbers are defined by some means. In **Asir** an algebraic number field is defined virtually as a number field obtained by adjoining a finite number of algebraic numbers to the rational number field.

A new algebraic number is introduced in **Asir** in such a way where it is defined as a root of an uni-variate polynomial whose coefficients include already defined algebraic numbers as well as rational numbers. We shall call such a newly defined algebraic number a **root**. Also, we call such an uni-variate polynomial the defining polynomial of that **root**.

```
[0] A0=newalg(x^2+1);
(#0)
[1] A1=newalg(x^3+A0*x+A0);
(#1)
[2]  [type(A0),ntype(A0)];
[1,2]
```

In this example, the algebraic number assigned to `A0` is defined as a **root** of a polynomial `x^2+1`; that of `A1` is defined as a **root** of a polynomial `x^3+A0*x+A0`, which you see contains the previously defined **root** (`A0`) in its coefficients.

The argument to `newalg()`, i.e., the defining polynomial, must satisfy the following conditions.

1. A defining polynomial must be an uni-variate polynomial.

2. A defining polynomial is used to simplify expressions containing that algebraic number. The procedure of such simplification is performed by an internal routine similar to the built-in function `srem()`, where the defining polynomial is used for the second argument, the divisor. By this reason, the leading coefficient of the defining polynomial must be a rational number (must not be an algebraic number.)

3. Every coefficients of a defining polynomial must be a '(multi-variate) polynomial' in already defined **root**'s. Here, '(multi-variate) polynomial' means a mathematical concept, not the object type 'polynomial' in **Asir**.

4. A defining polynomial must be irreducible over the field that is obtained by adjoining all **root**'s contained in its coefficients to the rational number field.

Only the first two conditions (1 and 2) are checked by function `newalg()`. Among all, it should be emphasized that no check is done for the irreducibility at all. The reason is that the irreducibility test requires enormously much computation time. You are trusted whether to check it at your own risk.

Once a **root** has been defined by `newalg()` function, it is given the type identifier for a number, and furthermore, the sub-type identifier for an algebraic number. (See Section 6.8.1 [type], page 76, Section 6.8.2 [ntype], page 77.) Also, any rational combination of rational numbers and **root**'s is an algebraic number.

```
[87] N=(A0^2+A1)/(A1^2-A0-1);
((#1+#0^2)/(#1^2-#0-1))
```

```
    [88] [type(N),ntype(N)];
    [1,2]
```

As you see it in the example, a **root** is displayed as `#n`. But, you cannot input that **root** in its immediate output form. You have to refer to a **root** by a program variable assigned to the **root**, or to get it by `alg(n)` function, or by several other indirect means. A strange use of `newalg()`, with a same argument polynomial (except for the name of its main variable), will yield the old **root** instead of a new **root** though it is apparently inefficient.

```
    [90] alg(0);
    (#0)
    [91] newalg(t^2+1);
    (#0)
```

The defining polynomial of a **root** can be obtained by `defpoly()` function.

```
    [96] defpoly(A0);
    t#0^2+1
    [97] defpoly(A1);
    t#1^3+t#0*t#1+t#0
```

Here, you see a strange expression, `t#0` and `t#1`. They are a specially indeterminates generated and maintained by **Asir** internally. Indeterminate `t#0` corresponds to **root** `#0`, and `t#0` to `#1`. These indeterminates also cannot be input directly by a user in their immediate forms. You can get them by several ways: by `var()` function, or `algv(n)` function.

```
    [98] var(@);
    t#1
    [99] algv(0);
    t#0
    [100]
```

9.2 Operations over algebraic numbers

In the previous section, we explained about the representation of algebraic numbers and their defining method. Here, we describe operations on algebraic numbers. Only a few functions are built-in, and almost all functions are provided as user defined functions. The file containing their definitions is 'sp', and it is placed under the same directory as 'gr' is placed, i.e., the standard library directory of **Asir**.

```
    [0] load("gr")$
    [1] load("sp")$
```

Or if you always need them, it is more convenient to include the `load` commands in '$HOME/.asirrc'.

Like the other numbers, algebraic numbers can get arithmetic operations applied. Simplification, however, by defining polynomials are not automatically performed. It is left to users to simplify their expressions. A fatal error shall result if the denominator expression will be simplified to 0. Therefore, be careful enough when you will create and deal with algebraic numbers which may denominators in their expressions.

Use `simpalg()` function for simplification of algebraic numbers by defining polynomials.

```
    [49] T=A0^2+1;
    (#0^2+1)
```

```
[50] simpalg(T);
0
```

Function `simpalg()` simplifies algebraic numbers which have the same structures as rational expressions in their appearances.

```
[39] A0=newalg(x^2+1);
(#0)
[40] T=(A0^2+A0+1)/(A0+3);
((#0^2+#0+1)/(#0+3))
[41] simpalg(T);
(3/10*#0+1/10)
[42] T=1/(A0^2+1);
((1)/(#0^2+1))
[43] simpalg(T);
div : division by 0
stopped in invalgp at line 258 in file "/usr/local/lib/asir/sp"
258                    return 1/A;
(debug)
```

This example shows an error caused by zero division in the course of program execution of `simpalg()`, which attempted to simplify an algebraic number expression of which the denominator is 0.

Function `simpalg()` also can take a polynomial as its argument and simplifies algebraic numbers in its coefficients.

```
[43] simpalg(1/A0*x+1/(A0+1));
(-#0)*x+(-1/2*#0+1/2)
```

Thus, you can operate in polynomials which contain algebraic numbers as you do usually in ordinary polynomials, except for proper simplification by `simpalg()`. You may sometimes feel needs to convert **root**'s into indeterminates, especially when you are working for norm computation in algorithms for algebraic factorization. Function `algptorat()` is used for such cases.

```
[83] A0=newalg(x^2+1);
(#0)
[84] A1=newalg(x^3+A0*x+A0);
(#1)
[85] T=(2*A0+A1*A0+A1^2)*x+(1+A1)/(2+A0);
(#1^2+#0*#1+2*#0)*x+((#1+1)/(#0+2))
[86] S=algptorat(T);
(((t#0+2)*t#1^2+(t#0^2+2*t#0)*t#1+2*t#0^2+4*t#0)*x+t#1+1)/(t#0+2)
[87] algptorat(coef(T,1));
t#1^2+t#0*t#1+2*t#0
```

As you see by the example, function `algptorat()` converts **root**'s, `#n`, in polynomials and numbers into its associated indeterminates, `t#n`. As was already mentioned those indeterminates cannot be directly input in their immediate form. The restriction is adopted to avoid the confusion that might happen if the user could input such internally generatable indeterminates.

The associated indeterminate to a **root** is reversely converted into the **root** by `rattoalgp()` function.

```
[88] rattoalgp(S,[alg(0)]);
(((#0+2)/(#0+2))*t#1^2+((#0^2+2*#0)/(#0+2))*t#1
+((2*#0^2+4*#0)/(#0+2)))*x+((1)/(#0+2))*t#1+((1)/(#0+2))
[89] rattoalgp(S,[alg(0),alg(1)]);
(((#0^3+6*#0^2+12*#0+8)*#1^2+(#0^4+6*#0^3+12*#0^2+8*#0)*#1
+2*#0^4+12*#0^3+24*#0^2+16*#0)/(#0^3+6*#0^2+12*#0+8))*x
+((((#0+2)*#1+#0+2)/(#0^2+4*#0+4))
[90] rattoalgp(S,[alg(1),alg(0)]);
(((#0+2)*#1^2+(#0^2+2*#0)*#1+2*#0^2+4*#0)/(#0+2))*x
+((#1+1)/(#0+2))
[91] simpalg(@89);
(#1^2+#0*#1+2*#0)*x+((-1/5*#0+2/5)*#1-1/5*#0+2/5)
[92] simpalg(@90);
(#1^2+#0*#1+2*#0)*x+((-1/5*#0+2/5)*#1-1/5*#0+2/5)
```

Function `rattoalgp()` takes as the second argument a list consisting of **root**'s that you want to convert, and converts them successively from the left. This example shows that apparent difference of the results due to the order of such conversion will vanish by simplification yielding the same result. Functions `algptorat()` and `rattoalgp()` can be conveniently used for your own simplification.

9.3 Representation of algebraic numbers by distributed polynomials

Simplification of algebraic numbers containing `root` is not done automatically and should be done by users. There is another representation of algebraic numbers, for which the results of fundamental operations are automatically simplified. This representations are designed so that operations are efficiently performed especially when the field is a successive extension and it can be used as a ground field for Groebner basis related functions. Internally an algebraic number of this type is defined as an object called `DAlg`. A `DAlg` is represented as a fraction. The denominator is an integer and the numerator is a distributed polynomial with integral coefficients.

`DAlg` is generated as an element of an algebraic number field set by `set_field()`. There are two methods to generate a `DAlg`. `algtodalg()` converts an algebraic number containing `root` to `DAlg`. `dptodalg()` directly converts a distributed polynomial to `DAlg`.

```
[0] A=newalg(x^2+1);
(#0)
[1] B=newalg(x^3+A*x+A);
(#1)
[2] set_field([B,A]);
0
[3] C=algtodalg(A+B);
((1)*<<1,0>>+(1)*<<0,1>>)
[4] C^5;
((-11)*<<2,1>>+(5)*<<2,0>>+(10)*<<1,1>>+(9)*<<1,0>>+(11)*<<0,1>>
```

```
+(-1)*<<0,0>>)
[5] 1/C;
((2)*<<2,1>>+(-1)*<<2,0>>+(1)*<<1,1>>+(2)*<<1,0>>+(-3)*<<0,1>>
+(-1)*<<0,0>>)/5
```

In this example Q(a,b) (a^2+1=0, b^3+ab+b=0) is set as the current ground field, and (a+b)^5 and 1/(a+b) are simplified in the field. The numerators of the results are printed as distributed polynomials.

9.4 Operations for uni-variate polynomials over an algebraic number field

In the file 'sp' are provided functions for uni-variate polynomial factorization and uni-variate polynomial GCD computation over algebraic numbers, and furthermore, as an application of them, functions to compute splitting fields of univariate polynomials.

9.4.1 GCD

Greatest common divisors (GCD) over algebraic number fields are computed by `cr_gcda()` function. This function computes GCD by using modular computation and Chinese remainder theorem and it works for the case where the ground field is a multiple extension.

```
[63] A=newalg(t^9-15*t^6-87*t^3-125);
(#0)
[64] B=newalg(75*s^2+(10*A^7-175*A^4-470*A)*s+3*A^8-45*A^5-261*A^2);
(#1)
[65] P1=75*x^2+(150*B+10*A^7-175*A^4-395*A)*x
+(75*B^2+(10*A^7-175*A^4-395*A)*B+13*A^8-220*A^5-581*A^2)$
[66] P2=x^2+A*x+A^2$
[67] cr_gcda(P1,P2);
27*x+((#0^6-19*#0^3-65)*#1-#0^7+19*#0^4+38*#0)
```

9.4.2 Square-free factorization and Factorization

For square-free factorization (of uni-variate polynomials over algebraic number fields), we employ the most fundamental algorithm which begins first to compute GCD of a polynomial and its derivative. The function to do this factorization is `asq()`.

```
[116] A=newalg(x^2+x+1);
(#4)
[117] T=simpalg((x+A+1)*(x^2-2*A-3)^2*(x^3-x-A)^2);
x^11+(#4+1)*x^10+(-4*#4-8)*x^9+(-10*#4-4)*x^8+(16*#4+20)*x^7
+(24*#4-6)*x^6+(-29*#4-31)*x^5+(-15*#4+28)*x^4+(38*#4+29)*x^3
+(#4-23)*x^2+(-21*#4-7)*x+(3*#4+8)
[118] asq(T);
[[x^5+(-2*#4-4)*x^3+(-#4)*x^2+(2*#4+3)*x+(#4-2),2],[x+(#4+1),1]]
```

Like factorization over the rational number field, the result is presented, commonly to both square-free factorization and factorization, as a list whose elements are pairs (list of two elements) in the form [**factor**, **multiplicity**] without the constant multiple part.

Here, it should be noticed that the products of all factors of the result may DIFFER from the input polynomial by a constant. The reason is that the factors are normalized so that they have integral leading coefficients for the sake of readability.

This incongruity may happen to square-free factorization and factorization commonly.

The algorithm employed for factorization over algebraic number fields is an improvement of the norm method by Trager. It is especially very effective to factorize a polynomial over a field obtained by adjoining some of its **root**'s to the base field.

```
[119] af(T,[A]);
[[x^3-x+(-#4),2],[x^2+(-2*#4-3),2],[x+(#4+1),1]]
```

The function takes two arguments: The second argument is a list of **root**'s. Factorization is performed over a field obtained by adjoining the **root**'s to the rational number field. It is important to keep in mind that the ordering of the **root**'s must obey a restriction: Last defined should come first. The automatic re-ordering is not done. It should be done by yourself.

The efficiency of factorization via norm depends on the efficiency of the norm computation and univariate factorization over the rationals. Especially the latter often causes combinatorial explosion and the computation will stick in such a case.

```
[120] B=newalg(x^2-2*A-3);
(#5)
[121] af(T,[B,A]);
[[x+(#5),2],[x^3-x+(-#4),2],[x+(-#5),2],[x+(#4+1),1]]
```

9.4.3 Splitting fields

This operation may be somewhat unusual and for specific interest. (Galois Group for example.) Procedurally, however, it is easy to obtain the splitting field of a polynomial by repeated application of algebraic factorization described in the previous section. The function is `sp()`.

```
[103] sp(x^5-2);
[[x+(-#1),2*x+(#0^3*#1^3+#0^4*#1^2+2*#1+2*#0),2*x+(-#0^4*#1^2),
2*x+(-#0^3*#1^3),x+(-#0)],
[[(#1),t#1^4+t#0*t#1^3+t#0^2*t#1^2+t#0^3*t#1+t#0^4],[(#0),t#0^5-2]]]
```

Function `sp()` takes only one argument. The result is a list of two element: The first element is a list of linear factors, and the second one is a list whose elements are pairs (list of two elements) in the form [**root**, `algptorat`(**defining polynomial**)]. The second element, a list of pairs of form [**root**,`algptorat`(**defining polynomial**)], corresponds to the **root**'s which are adjoined to eventually obtain the splitting field. They are listed in the reverse order of adjoining. Each of the defining polynomials in the list is, of course, guaranteed to be irreducible over the field obtained by adjoining all **root**'s defined before it.

The first element of the result, a list of linear factors, contains all irreducible factors of the input polynomial over the field obtained by adjoining all **root**'s in the second element of the result. Because such field is the splitting field of the input polynomial, factors in the result are all linear as the consequence.

Similarly to function `af()`, the product of all resulting factors may yield a polynomial which differs by a constant.

9.5 Summary of functions for algebraic numbers

9.5.1 `newalg`

`newalg(`*defpoly*`)`
          :: Creates a new **root**.

return   algebraic number (**root**)

defpoly   polynomial

- Creates a new **root** (algebraic number) with its defining polynomial *defpoly*.
- For constraints on *defpoly*, See Section 9.1 [Representation of algebraic numbers], page 162.

      [0] A0=newalg(x^2-2);
      (#0)

Reference
          Section 9.5.2 [`defpoly`], page 168

9.5.2 `defpoly`

`defpoly(`*alg*`)`
          :: Returns the defining polynomial of **root** *alg*.

return   polynomial

alg   algebraic number (`root`)

- Returns the defining polynomial of **root** *alg*.
- If the argument *alg*, a **root**, is `#n`, then the main variable of its defining polynomial is `t#n`.

      [1] defpoly(A0);
      t#0^2-2

Reference
          Section 9.5.1 [`newalg`], page 168, Section 9.5.3 [`alg`], page 168, Section 9.5.4 [`algv`], page 169

9.5.3 `alg`

`alg(`*i*`)`   :: Returns a **root** which correspond to the index *i*.

return   algebraic number (**root**)

i   integer

- Returns `#i`, a **root**.
- Because `#i` cannot be input directly, this function provides an alternative way: input `alg(`*i*`)`.

      [2] x+#0;
      syntax error
      0
      [3] alg(0);

```
(#0)
```

9.5.4 algv

algv(*i*)   :: Returns the associated indeterminate with alg(*i*).

return   polynomial

i   integer

- Returns an indeterminate t#*i*
- Since indeterminate t#*i* cannot be input directly, it is input by algv(*i*).

```
[4] var(defpoly(A0));
t#0
[5] t#0;
syntax error
0
[6] algv(0);
t#0
```

9.5.5 simpalg

simpalg(*rat*)
          :: Simplifies algebraic numbers in a rational expression.

return   rational expression

rat   rational expression

- Defined in the file 'sp'.
- Simplifies algebraic numbers contained in numbers, polynomials, and rational expressions by the defining polynomials of **root**'s contained in them.
- If the argument is a number having the denominator, it is rationalized and the result is a polynomial in **root**'s.
- If the argument is a polynomial, each coefficient is simplified.
- If the argument is a rational expression, its denominator and numerator are simplified as a polynomial.

```
[7] simpalg((1+A0)/(1-A0));
simpalg undefined
return to toplevel
[7] load("sp")$
[46] simpalg((1+A0)/(1-A0));
(-2*#0-3)
[47] simpalg((2-A0)/(2+A0)*x^2-1/(3+A0));
```

```
(-2*#0+3)*x^2+(1/7*#0-3/7)
[48] simpalg((x+1/(A0-1))/(x-1/(A0+1)));
(x+(#0+1))/(x+(-#0+1))
```

9.5.6 `algptorat`

`algptorat(`*poly*`)`

:: Substitutes the associated indeterminate for every **root**

return   polynomial

poly   polynomial

- Defined in the file 'sp'.

- Substitutes the associated indeterminate `t#`*n* for every **root** `#`*n* in a polynomial.

    ```
    [49] algptorat((-2*alg(0)+3)*x^2+(1/7*alg(0)-3/7));
    (-2*t#0+3)*x^2+1/7*t#0-3/7
    ```

`Reference`

Section 9.5.2 [`defpoly`], page 168, Section 9.5.4 [`algv`], page 169

9.5.7 `rattoalgp`

`rattoalgp(`*poly*`,`*alglist*`)`

:: Substitutes a **root** for the associated indeterminate with the **root**.

return   polynomial

poly   polynomial

alglist   list

- Defined in the file 'sp'.

- The second argument is a list of **root**'s. Function `rattoalgp()` substitutes a **root** for the associated indeterminate of the **root**.

    ```
    [51] rattoalgp((-2*algv(0)+3)*x^2+(1/7*algv(0)-3/7),[alg(0)]);
    (-2*#0+3)*x^2+(1/7*#0-3/7)
    ```

`Reference`

Section 9.5.3 [`alg`], page 168, Section 9.5.4 [`algv`], page 169

9.5.8 `cr_gcda`

`cr_gcda(`*poly1*`,`*poly2*`)`

:: GCD of two uni-variate polynomials over an algebraic number field.

return   polynomial

poly1 poly2

polynomial

- Defined in the file 'sp'.

- Finds the GCD of two uni-variate polynomials.

```
[76] X=x^6+3*x^5+6*x^4+x^3-3*x^2+12*x+16$
[77] Y=x^6+6*x^5+24*x^4+8*x^3-48*x^2+384*x+1024$
[78] A=newalg(X);
(#0)
[79] cr_gcda(X,subst(Y,x,x+A));
x+(-#0)
```

Reference
        Section 8.10.1 [gr hgr gr_mod], page 131, Section 9.5.10 [asq af af_noalg],
        page 171

9.5.9 sp_norm

sp_norm(*alg*,*var*,*poly*,*alglist*)
        :: Norm computation over an algebraic number field.

return   polynomial

var   The main variable of *poly*

poly   univariate polynomial

alg   root

alglist   root list

- Defined in the file 'sp'.
- Computes the norm of *poly* with respect to *alg*. Namely, if we write $\mathbf{K} = \mathbf{Q}(alglist \setminus \{alg\})$, The function returns a product of all conjugates of *poly*, where the conjugate of polynomial *poly* is a polynomial in which the algebraic number *alg* is substituted for its conjugate over $\mathbf{K}$.
- The result is a polynomial over $\mathbf{K}$.
- The method of computation depends on the input. Currently direct computation of resultant and Chinese remainder theorem are used but the selection is not necessarily optimal. By setting the global variable USE_RES to 1, the builtin function res() is always used.

```
[0] load("sp")$
[39] A0=newalg(x^2+1)$
[40] A1=newalg(x^2+A0)$
[41] sp_norm(A1,x,x^3+A0*x+A1,[A1,A0]);
x^6+(2*#0)*x^4+(#0^2)*x^2+(#0)
[42] sp_norm(A0,x,@@,[A0]);
x^12+2*x^8+5*x^4+1
```

Reference
        Section 6.3.14 [res], page 54, Section 9.5.10 [asq af af_noalg], page 171

9.5.10 asq, af, af_noalg

asq(*poly*)
        :: Square-free factorization of polynomial *poly* over an algebraic number field.

`af(`*poly*`,`*alglist*`)`
`af_noalg(`*poly*`,`*defpolylist*`)`

     :: Factorization of polynomial *poly* over an algebraic number field.

return   list

poly   polynomial

alglist   `root` list

defpolylist   `root` list of pairs of an indeterminate and a polynomial

- Both defined in the file '`sp`'.
- If the inputs contain no **root**'s, these functions run fast since they invoke functions over the integers. In contrast to this, if the inputs contain **root**'s, they sometimes take a long time, since `cr_gcda()` is invoked.
- Function `af()` requires the specification of base field, i.e., list of **root**'s for its second argument.
- In the second argument `alglist`, **root** defined last must come first.
- In `af(F,AL)`, `AL` denotes a list of `roots` and it represents an algebraic number field. In `AL=[An,...,A1]` each `Ak` should be defined as a root of a defining polynomial whose coefficients are in `Q(A(k+1),...,An)`.

```
[1] A1 = newalg(x^2+1);
[2] A2 = newalg(x^2+A1);
[3] A3 = newalg(x^2+A2*x+A1);
[4] af(x^2+A2*x+A1,[A2,A1]);
[[x^2+(#1)*x+(#0),1]]
```

To call `sp_noalg`, one should replace each algebraic number *ai* in *poly* with an indeterminate *vi*. `defpolylist` is a list [[vn,dn(vn,...,v1)],...,[v1,d(v1)]]. In this expression *di*(vi,...,v1) is a defining polynomial of *ai* represented as a multivariate polynomial.

```
[1] af_noalg(x^2+a2*x+a1,[[a2,a2^2+a1],[a1,a1^2+1]]);
[[x^2+a2*x+a1,1]]
```

- The result is a list, as a result of usual factorization, whose elements is of the form [**factor**, **multiplicity**]. In the result of `af_noalg`, algebraic numbers in factor are replaced by the indeterminates according to *defpolylist*.
- The product of all factors with multiplicities counted may differ from the input polynomial by a constant.

```
[98] A = newalg(t^2-2);
(#0)
[99] asq(-x^4+6*x^3+(2*alg(0)-9)*x^2+(-6*alg(0))*x-2);
[[-x^2+3*x+(#0),2]]
[100] af(-x^2+3*x+alg(0),[alg(0)]);
[[x+(#0-1),1],[-x+(#0+2),1]]
[101] af_noalg(-x^2+3*x+a,[[a,x^2-2]]);
[[x+a-1,1],[-x+a+2,1]]
```

`Reference`

     Section 9.5.8 [`cr_gcda`], page 170, Section 6.3.15 [`fctr sqfr`], page 54

9.5.11 `sp`, `sp_noalg`

`sp(`*poly*`)`
`sp_noalg(`*poly*`)`
>   :: Finds the splitting field of polynomial *poly* and splits.

return   list

poly   polynomial

- Defined in the file 'sp'.
- Finds the splitting field of *poly*, an uni-variate polynomial over with rational coefficients, and splits it into its linear factors over the field.
- The result consists of a two element list: The first element is the list of all linear factors of *poly*; the second element is a list which represents the successive extension of the field. In the result of `sp_noalg` all the algebraic numbers are replaced by the special indeterminate associated with it, that is `t#i` for `#i`. By this operation the result of `sp_noalg` is a list containing only integral polynomials.
- The splitting field is represented as a list of pairs of form `[root,`
  `algptorat(defpoly(root))]`.   In more detail, the list is interpreted as a representation of successive extension obtained by adjoining **root**'s to the rational number field. Adjoining is performed from the right **root** to the left.
- `sp()` invokes `sp_norm()` internally. Computation of norm is done by several methods according to the situation but the algorithm selection is not always optimal and a simple resultant computation is often superior to the other methods. By setting the global variable `USE_RES` to 1, the builtin function `res()` is always used.

```
[101] L=sp(x^9-54);
[[x+(-#2),-54*x+(#1^6*#2^4),54*x+(#1^6*#2^4+54*#2),
54*x+(-#1^8*#2^2),-54*x+(-#1^5*#2^5),54*x+(#1^5*#2^5+#1^8*#2^2),
-54*x+(-#1^7*#2^3-54*#1),54*x+(-#1^7*#2^3),x+(-#1)],
[[(#2),t#2^6+t#1^3*t#2^3+t#1^6],[(#1),t#1^9-54]]]
[102] for(I=0,M=1;I<9;I++)M*=L[0][I];
[111] M=simpalg(M);
-1338925209984*x^9+72301961339136
[112] ptozp(M);
-x^9+54
```

Reference
>   Section 9.5.10 [`asq af af_noalg`], page 171, Section 9.5.2 [`defpoly`], page 168,
>   Section 9.5.6 [`algptorat`], page 170, Section 9.5.9 [`sp_norm`], page 171.

9.5.12 `set_field`

`set_field(`*rootlist*`)`
>   :: Set an algebraic number field as the currernt ground field.

return   0

rootlist   A list of `root`

- `set_field()` sets an algebraic number field generated by `root` in *rootlist* over Q.

- You don't care about the order of `root` in *rootlist*, because `root` are automatically ordered internally.

```
[0] A=newalg(x^2+1);
(#0)
[1] B=newalg(x^3+A);
(#1)
[2] C=newalg(x^4+B);
(#1)
[3] set_field([C,B,A]);
0
```

```
Reference
```

9.5.13 algtodalg, dalgtoalg, dptodalg, dalgtodp

`algtodalg(alg)`
          :: Converts an algebraic number *alg* to a `DAlg`.

`dalgtoalg(dalg)`
          :: Converts a `DAlg` `dalg` to an algebraic number.

`dptodalg(dp)`
          :: Converts an algebraic number *alg* to a `DAlg`.

`dalgtodp(dalg)`
          :: Converts a `DAlg` `dalg` to an algebraic number.

return   An algebraic number, a `DAlg` or a list [distributed polynomial,denominator]

alg   an algebraic number containing `root`

dp   a distributed polynomial over Q

- These functions are converters between `DAlg` and an algebraic number containing `root`, or a distributed polynomial.
- A ground field to which a `DAlg` belongs must be set by `set_field()` in advance.
- `dalgtodp()` returns a list containing the numerator (a distributed polynomial) and the denominator (an integer).
- `algtodalg()`, `dptodalg()` return the simplified result.

```
[0] A=newalg(x^2+1);
(#0)
[1] B=newalg(x^3+A*x+A);
(#1)
[2] set_field([B,A]);
0
[3] C=algtodalg((A+B)^10);
((408)*<<2,1>>+(103)*<<2,0>>+(-36)*<<1,1>>+(-446)*<<1,0>>
+(-332)*<<0,1>>+(-218)*<<0,0>>)
[4] dalgtoalg(C);
((408*#0+103)*#1^2+(-36*#0-446)*#1-332*#0-218)
```

```
[5] D=dptodalg(<<10,10>>/10+2*<<5,5>>+1/3*<<0,0>>);
((-9)*<<2,1>>+(57)*<<2,0>>+(-63)*<<1,1>>+(-12)*<<1,0>>
+(-60)*<<0,1>>+(1)*<<0,0>>)/30
[6] dalgtodp(D);
[(-9)*<<2,1>>+(57)*<<2,0>>+(-63)*<<1,1>>+(-12)*<<1,0>>
+(-60)*<<0,1>>+(1)*<<0,0>>,30]
```

Reference

Section 9.5.12 [set_field], page 173

10 Finite fields

10.1 Representation of finite fields

On **Asir** GF(p), GF(2^n), GF(p^n) can be defined, where GF(p) is a finite prime field of characteristic p, GF(2^n) is a finite field of characteristic 2 and GF(p^n) is a finite extension of GF(p). These are all defined by `setmod_ff()`.

```
[0] P=pari(nextprime,2^50);
1125899906842679
[1] setmod_ff(P);
1125899906842679
[2] field_type_ff();
1
[3] load("fff");
1
[4] F=defpoly_mod2(50);
x^50+x^4+x^3+x^2+1
[5] setmod_ff(F);
x^50+x^4+x^3+x^2+1
[6] field_type_ff();
2
[7] setmod_ff(x^3+x+1,1125899906842679);
[1*x^3+1*x+1,1125899906842679]
[8] field_type_ff();
3
[9] setmod_ff(3,5);
[3,x^5+2*x+1,x]
[10] field_type_ff();
4
```

If p is a positive integer, `setmod_ff(p)` sets GF(p) as the current base field. If f is a univariate polynomial of degree n, `setmod_ff(f)` sets GF(2^n) as the current base field. GF(2^n) is represented as an algebraic extension of GF(2) with the defining polynomial f *mod 2*. Furthermore, finite extensions of prime finite fields can be defined. See Section 3.2 [Types of numbers], page 14. In all cases the primality check of the argument is not done and the caller is responsible for it.

Correctly speaking there is no actual object corresponding to a 'base field'. Setting a base field means that operations on elements of finite fields are done according to the arithmetics of the base field. Thus, if operands of an arithmetic operation are both rational numbers, then the result is also a rational number. However, if one of the operands is in a finite field, then the other is automatically regarded as in the same finite field and the operation is done in the finite field.

A non zero element of a finite field belongs to the number and has object identifier 1. Its number identifier is 6 if the finite field is GF(p), 7 if it is GF(2^n).

There are several methods to input an element of a finite field. An element of GF(p) can be input by `simp_ff()`.

```
[0] P=pari(nextprime,2^50);
1125899906842679
[1] setmod_ff(P);
1125899906842679
[2] A=simp_ff(2^100);
3025
[3] ntype(@@);
6
```

In the case of GF(2^n) the following methods are available.

```
[0] setmod_ff(x^50+x^4+x^3+x^2+1);
x^50+x^4+x^3+x^2+1
[1] A=@;
(@)
[2] ptogf2n(x^50+1);
(@^50+1)
[3] simp_ff(@@);
(@^4+@^3+@^2)
[4] ntogf2n(2^10-1);
(@^9+@^8+@^7+@^6+@^5+@^4+@^3+@^2+@+1)
```

Elements of finite fields are numbers and one can apply field arithmetics to them. @ is a generator of GF(2^n) over GF(2). See Section 3.2 [Types of numbers], page 14.

10.2 Univariate polynomials on finite fields

In 'fff' square-free factorization, DDF (distinct degree factorization), irreducible factorization and primality check are implemented for univariate polynomials over finite fields.

Factorizers return lists of [**factor**,**multiplicity**]. The factor part is monic and the information on the leading coefficient of the input polynomial is abandoned.

The algorithm used in square-free factorization is the most primitive one.

The irreducible factorization proceeds as follows.

1. DDF

2. Nullspace computation by Berlekamp algorithm

3. Root finding of minimal polynomials of bases of the nullspace

4. Separation of irreducible factors by the roots

10.3 Polynomials on small finite fields

A multivariate polynomial over small finite field set by `setmod_ff(p,n)` can be factorized by using a builtin function `sffctr()`. `modfctr()` also factorizes a polynomial over a finite prime field. Internally, `modfctr()` creates a sufficiently large field extension of the specified ground field, and it calls `sffctr()`, then it constructs irreducible factors over the ground field from the factors returned by `sffctr()`.

10.4 Elliptic curves on finite fields

Several fundamental operations on elliptic curves over finite fields are provided as built-in functions.

An elliptic curve is specified by a vector [a b] of length 2, where a, b are elements of finite fields. If the current base field is a prime field, then [a b] represents y^2=x^3+ax+b. If the current base field is a finite field of characteristic 2, then [a b] represents y^2+xy=x^3+ax^2+b.

Points on an elliptic curve together with the point at infinity forms an additive group. The addition, the subtraction and the additive inverse operation are provided as `ecm_add_ff()`, `ecm_sub_ff()` and `ecm_chsgn_ff()` respectively. Here the representation of points are as follows.

- 0 denotes the point at infinity.
- The other points are represented by vectors [x y z] of length 3 with non-zero z.

[x y z] represents a projective coordinate and it corresponds to [x/z y/z] in the affine coordinate. To apply the above operations to a point [x y], [x y 1] should be used instead as an argument. The result of an operation is also represented by the projective coordinate. As the third coordinate is not always equal to 1, one has to divide the first and the second coordinate by the third one to obtain the affine coordinate.

10.5 Functions for Finite fields

10.5.1 `setmod_ff`

```
setmod_ff([p|defpoly2])
setmod_ff([defpolyp,p])
setmod_ff([p,n])
```
          :: Sets/Gets the current base fields.

return	number or polynomial
p	prime
defpoly2	univariate polynomial irreducible over GF(2)
defpolyp	univariate polynomial irreducible over GF(p)
n	the extension degree

- If the argument is a non-negative integer p, GF(p) is set as the current base field.
- If the argument is a polynomial *defpoly2*, GF(2^deg(*defpoly2* mod 2)) = GF(2)[t]/(*defpoly2*(t) mod2) is set as the current base field.
- If the arguments are a polynomial *defpolyp* and a prime p, GF(p^deg(*defpolyp*)) = GF(p)[t]/(*defpolyp*(t)) is set as the current base field.
- If the arguments are a prime p and an extension degree n, GF(p^n) is set as the current base field. p^n must be less than 2^29 and if p is greater than or equal to 2^14, then n must be equal to 1.
- If no argument is specified, the modulus indicating the current base field is returned. If the current base field is GF(p), p is returned. If it is GF(2^n), the defining polynomial is

returned. If it is GF(p^n) defined by `setmod_ff(`*defpoly*`,p)`, [*defpolyp*,*p*] is returned. If it is GF(p^n) defined by `setmod_ff(p,n)`, [*p*,*defpoly*,*prim_elem*] is returned. Here, *defpoly* is the defining polynomial of the n-th extension, and *prim_elem* is the generator of the multiplicative group of GF(p^n).

- Any irreducible univariate polynomial over GF(2) is available to set GF(2^n). However the use of `defpoly_mod2()` is recommended for efficiency.

```
[174] defpoly_mod2(100);
x^100+x^15+1
[175] setmod_ff(@@);
x^100+x^15+1
[176] setmod_ff();
x^100+x^15+1
[177] setmod_ff(x^4+x+1,547);
[1*x^4+1*x+1,547]
[178] setmod_ff(2,5);
[2,x^5+x^2+1,x]
```

References
          Section 10.5.14 [`defpoly_mod2`], page 185

10.5.2 `field_type_ff`

`field_type_ff()`
          :: Type of the current base field.

return   integer

- Returns the type of the current base field.
- If no field is set, 0 is returned. If GF(p) is set, 1 is returned. If GF(2^n) is set, 2 is returned.

```
[0] field_type_ff();
0
[1] setmod_ff(3);
3
[2] field_type_ff();
1
[3] setmod_ff(x^2+x+1);
x^2+x+1
[4] field_type_ff();
2
```

References
          Section 10.5.1 [`setmod_ff`], page 178

10.5.3 `field_order_ff`

`field_order_ff()`
          :: Order of the current base field.

return   integer

- Returns the order of the current base field.
- q is returned if the current base field is GF(q).

```
[0] field_order_ff();
field_order_ff : current_ff is not set
return to toplevel
[0] setmod_ff(3);
3
[1] field_order_ff();
3
[2] setmod_ff(x^2+x+1);
x^2+x+1
[3] field_order_ff();
4
```

References
        Section 10.5.1 [setmod_ff], page 178

10.5.4 characteristic_ff

```
characteristic_ff()
```
        :: Characteristic of the current base field.

return   integer
- Returns the characteristic of the current base field.
- p is returned if GF(p), where p is a prime, is set. *2* is returned if GF(2^n) is set.

```
[0] characteristic_ff();
characteristic_ff : current_ff is not set
return to toplevel
[0] setmod_ff(3);
3
[1] characteristic_ff();
3
[2] setmod_ff(x^2+x+1);
x^2+x+1
[3] characteristic_ff();
2
```

References
        Section 10.5.1 [setmod_ff], page 178

10.5.5 extdeg_ff

```
extdeg_ff()
```
        :: Extension degree of the current base field over the prime field.

return   integer
- Returns the extension degree of the current base field over the prime field.
- 1 is returned if GF(p), where p is a prime, is set. n is returned if GF(2^n) is set.

```
[0] extdeg_ff();
extdeg_ff : current_ff is not set
return to toplevel
[0] setmod_ff(3);
3
[1] extdeg_ff();
1
[2] setmod_ff(x^2+x+1);
x^2+x+1
[3] extdeg_ff();
2
```

`References`
> Section 10.5.1 [`setmod_ff`], page 178

10.5.6 `simp_ff`

`simp_ff(obj)`
> :: Converts numbers or coefficients of polynomials into elements in finite fields.

return   number or polynomial

obj   number or polynomial

- Converts numbers or coefficients of polynomials into elements in finite fields.

- It is used to convert integers or intrgral polynomials int elements of finite fields or polynomials over finite fields.

- An element of a finite field may not have the reduced representation. In such case an application of `simp_ff` ensures that the output has the reduced representation. If a small finite field is set as a ground field, an integer is projected the finite prime field, then it is embedded into the ground field. `ptosfp()` can be used for direct projection to the ground field.

```
[0] simp_ff((x+1)^10);
x^10+10*x^9+45*x^8+120*x^7+210*x^6+252*x^5+210*x^4+120*x^3+45*x^2+10*x+1
[1] setmod_ff(3);
3
[2] simp_ff((x+1)^10);
1*x^10+1*x^9+1*x+1
[3] ntype(coef(@@,10));
6
[4] setmod_ff(2,3);
[2,x^3+x+1,x]
[5] simp_ff(1);
@_0
[6] simp_ff(2);
0
[7] ptosfp(2);
@_1
```

References
          Section 10.5.1 [setmod_ff], page 178, Section 10.5.8 [lmptop], page 182,
          Section 10.5.10 [gf2nton], page 183, Section 10.5.13 [ptosfp sfptop], page 185

10.5.7 random_ff

random_ff()
          :: Random generation of an element of a finite field.

return   element of a finite field

- Generates an element of the current base field randomly.
- The same random generator as in random(), lrandom() is used.

```
[0] random_ff();
random_ff : current_ff is not set
return to toplevel
[0] setmod_ff(pari(nextprime,2^40));
1099511627791
[1] random_ff();
561856154357
[2] random_ff();
45141628299
```

References
          Section 10.5.1 [setmod_ff], page 178, Section 6.1.8 [random], page 38,
          Section 6.1.9 [lrandom], page 38

10.5.8 lmptop

lmptop(*obj*)
          :: Converts the coefficients of a polynomial over $GF(p)$ into integers.

return   integral polynomial

obj   polynomial over $GF(p)$

- Converts the coefficients of a polynomial over $GF(p)$ into integers.
- An element of $GF(p)$ is represented by a non-negative integer r less than p. Each coefficient of a polynomial is converted into an integer object whose value is r.

```
[0] setmod_ff(pari(nextprime,2^40));
1099511627791
[1] F=simp_ff((x-1)^10);
1*x^10+1099511627781*x^9+45*x^8+1099511627671*x^7+210*x^6
+1099511627539*x^5+210*x^4+1099511627671*x^3+45*x^2+1099511627781*x+1
[2] setmod_ff(547);
547
[3] F=simp_ff((x-1)^10);
1*x^10+537*x^9+45*x^8+427*x^7+210*x^6+295*x^5+210*x^4+427*x^3
+45*x^2+537*x+1
[4] lmptop(F);
x^10+537*x^9+45*x^8+427*x^7+210*x^6+295*x^5+210*x^4+427*x^3
```

```
    +45*x^2+537*x+1
    [5] lmptop(coef(F,1));
    537
    [6] ntype(@@);
    0
```

References
      Section 10.5.6 [`simp_ff`], page 181

10.5.9 `ntogf2n`

`ntogf2n(`*m*`)`
      :: Converts a non-negative integer into an element of GF(2^*n*).

return   element of GF(2^*n*)

m   non-negative integer

- Let *m* be a non-negative integer. *m* has the binary representation *m=m0+m1*2+...+mk*2^k*. This function returns an element of GF(2^*n*) = GF(2)[t]/(g(t)), *m0+m1*t+...+mk*t^k* mod g(t).
- Apply `simp_ff()` to reduce the result.

```
    [1] setmod_ff(x^30+x+1);
    x^30+x+1
    [2] N=ntogf2n(2^100);
    (@^100)
    [3] simp_ff(N);
    (@^13+@^12+@^11+@^10)
```

References
      Section 10.5.10 [`gf2nton`], page 183

10.5.10 `gf2nton`

`gf2nton(`*m*`)`
      :: Converts an element of GF(2^*n*) into a non-negative integer.

return   non-negative integer

m   element of GF(2^*n*)

- The inverse of `gf2nton`.

```
    [1] setmod_ff(x^30+x+1);
    x^30+x+1
    [2] N=gf2nton(2^100);
    (@^100)
    [3] simp_ff(N);
    (@^13+@^12+@^11+@^10)
    [4] gf2nton(N);
    126765060022822940149670320537
    [5] gf2nton(simp_ff(N));
```

```
      15360
```

References
          Section 10.5.10 [gf2nton], page 183

10.5.11 `ptogf2n`

`ptogf2n(poly)`
          :: Converts a univariate polynomial into an element of GF(2^n).

return   element of GF(2^n)

poly   univariate polynomial

- Generates an element of GF(2^n) represented by *poly*. The coefficients are reduced modulo 2. The output is equal to the result by substituting @ for the variable of *poly*.

  ```
  [1] setmod_ff(x^30+x+1);
  x^30+x+1
  [2] ptogf2n(x^100);
  (@^100)
  ```

References
          Section 10.5.12 [gf2ntop], page 184

10.5.12 `gf2ntop`

`gf2ntop(m[,v])`
          :: Converts an element of GF(2^n) into a polynomial.

return   univariate polynomial

m   an element of GF(2^n)

v   indeterminate

- Returns a polynomial representing *m*.

- If *v* is used as the variable of the output. If *v* is not specified, the variable of the argument of the latest `ptogf2n()` call. The default variable is `x`.

  ```
  [1] setmod_ff(x^30+x+1);
  x^30+x+1
  [2] N=simp_ff(gf2ntop(2^100));
  (@^13+@^12+@^11+@^10)
  [5] gf2ntop(N);
  [207] gf2ntop(N);
  x^13+x^12+x^11+x^10
  [208] gf2ntop(N,t);
  t^13+t^12+t^11+t^10
  ```

References
          Section 10.5.11 [ptogf2n], page 184

10.5.13 `ptosfp, sfptop`

`ptosfp(`*p*`)`
`sfptop(`*p*`)`
                :: Transformation to/from a small finite field

return   polynomial

p   polynomial
  - `ptosfp()` converts coefficients of a polynomial to elements in a small finite field
    GF(*p*^*n*) set as a ground field. If a coefficient is already an element of the field,
    no conversion is done. If a coefficient is a positive integer, then its residue modulo
    p^*n* is expanded as *p*-adic integer, then *p* is substituted by *x*, finally the polynomial
    is converted to its corresponding logarithmic representation with respect to the
    primitive element. For example, GF(3^5) is represented as F(3)[*x*]/(*x^5+2*x+1*), and
    each element of the field is represented as @_*k* by its exponent *k* with respect to the
    primitive element *x*. *23 = 2*3^2+3+2* is represented as *2*x^2+x+2* and it is equivalent
    to *x^17* modulo *x^5+2*x+1*. Therefore an integer *23* is conterted to *@_17*.

  - `sfptop()` is the inverse of `ptosfp()`.

    ```
    [196] setmod_ff(3,5);
    [3,x^5+2*x+1,x]
    [197] A = ptosfp(23);
    @_17
    [198] 9*2+3+2;
    23
    [199] x^17-(2*x^2+x+2);
    x^17-2*x^2-x-2
    [200] sremm(@,x^5+2*x+1,3);
    0
    [201] sfptop(A);
    23
    ```

`References`
        Section 10.5.1 [`setmod_ff`], page 178, Section 10.5.6 [`simp_ff`], page 181

10.5.14 `defpoly_mod2`

`defpoly_mod2(`*d*`)`
            :: Generates an irreducible univariate polynomial over GF(2).

return   univariate polynomial

d   positive integer
  - Defined in 'fff'.

  - An irreducible univariate polynomial of degree *d* is returned.

  - If an irreducible trinomial *x^d+x^m+1* exists, then the one with the smallest *m* is re-
    turned. Otherwise, an irreducible pentanomial *x^d+x^m1+x^m2+x^m3+1* (*m1>m2>m3*)
    is returned. *m1*, *m2* and *m3* are determined as follows: Fix *m1* as small as possible.
    Then fix *m2* as small as possible. Then fix *m3* as small as possible.

References
    Section 10.5.1 [`setmod_ff`], page 178

10.5.15 `sffctr`

`sffctr(`*poly*`)`
    :: Irreducible factorization over a small finite field.

return   list

poly   polynomial over a finite field

- Factorize *poly* into irreducible factors over a small finite field currently set.
- The result is a list [[*f1,m1*],[*f2,m2*],...], where *fi* is a monic irreducible factor and *mi* is its multiplicity.

      [0] setmod_ff(2,10);
      [2,x^10+x^3+1,x]
      [1] sffctr((z*y^3+z*y)*x^3+(y^5+y^3+z*y^2+z)*x^2+z^11*y*x+z^10*y^3+z^11);█
      [[@_0,1],[@_0*z*y*x+@_0*y^3+@_0*z,1],[(@_0*y+@_0)*x+@_0*z^5,2]]

References
    Section 10.5.1 [`setmod_ff`], page 178, Section 6.3.17 [`modfctr`], page 56

10.5.16 `fctr_ff`

`fctr_ff(`*poly*`)`
    :: Irreducible univariate factorization over a finite field.

return   list

poly   univariate polynomial over a finite field

- Defined in '`fff`'.
- Factorize *poly* into irreducible factors over the current base field.
- The result is a list [[*f1,m1*],[*f2,m2*],...], where *fi* is a monic irreducible factor and *mi* is its multiplicity.
- The leading coefficient of *poly* is abandoned.

      [178] setmod_ff(2^64-95);
      18446744073709551521
      [179] fctr_ff(x^5+x+1);
      [[1*x+14123390394564558010,1],[1*x+6782485570826905238,1],
      [1*x+15987612182027639793,1],[1*x^2+1*x+1,1]]

References
    Section 10.5.1 [`setmod_ff`], page 178

10.5.17 `irredcheck_ff`

`irredcheck_ff(`*poly*`)`
    :: Primality check of a univariate polynomial over a finite field.

return   0|1

poly   univariate polynomial over a finite field

- Defined in 'fff'.
- Returns 1 if *poly* is irreducible over the current base field. Returns 0 otherwise.

  ```
  [178] setmod_ff(2^64-95);
  18446744073709551521
  [179] ] F=x^10+random_ff();
  x^10+14687973587364016969
  [180] irredcheck_ff(F);
  1
  ```

References
        Section 10.5.1 [setmod_ff], page 178

10.5.18 randpoly_ff

randpoly_ff(*d*, *v*)
        :: Generation of a random univariate polynomial over a finite field.

return   polynomial

d   positive integer

v   indeterminate

- Defined in 'fff'.
- Generates a polynomial of *v* such that the degree is less than *d* and the coefficients are in the current base field. The coefficients are generated by random_ff().

  ```
  [178] setmod_ff(2^64-95);
  18446744073709551521
  [179] ] F=x^10+random_ff();
  [180] randpoly_ff(3,x);
  17135261454578964298*x^2+4766826699653615429*x+18317369440429479651
  [181] randpoly_ff(3,x);
  7565988813172050604*x^2+7430075767279665339*x+4699662986224873544
  [182] randpoly_ff(3,x);
  10247781277095450395*x^2+10243690944992524936*x+4063829049268845492
  ```

References
        Section 10.5.1 [setmod_ff], page 178, Section 10.5.7 [random_ff], page 182

10.5.19 ecm_add_ff, ecm_sub_ff, ecm_chsgn_ff

ecm_add_ff(*p1*, *p2*, *ec*)
ecm_sub_ff(*p1*, *p2*, *ec*)
ecm_chsgn_ff(*p1*)
        :: Addition, Subtraction and additive inverse for points on an elliptic curve.

return   vector or 0

p1 p2   vector of length 3 or 0

ec   vector of length 2

- Let *p1*, *p2* be points on the elliptic curve represented by *ec* over the current base field. ecm_add_ff(*p1*,*p2*,*ec*), ecm_sub_ff(*p1*,*p2*,*ec*) and ecm_chsgn_ff(*p1*) returns *p1+p2*, *p1-p2* and *-p1* respectively.

- If the current base field is a prime field of odd order, then *ec* represents y^2=x^3+ec[0]x+ec[1]. If the characteristic of the current base field is 2, then *ec* represents y^2+xy=x^3+ec[0]x^2+ec[1].

- The point at infinity is represented by 0.

- If an argument denoting a point is a vector of length 3, then it is the projective coordinate. In such a case the third coordinate must not be 0.

- If the result is a vector of length 3, then the third coordinate is not equal to 0 but not necessarily 1. To get the result by the affine coordinate, the first and the second coordinates should be divided by the third coordinate.

- The check whether the arguments are on the curve is omitted.

```
[0] setmod_ff(1125899906842679)$
[1] EC=newvect(2,[ptolmp(1),ptolmp(1)])$
[2] Pt1=newvect(3,[1,-412127497938252,1])$
[3] Pt2=newvect(3,[6,-252647084363045,1])$
[4] Pt3=ecm_add_ff(Pt1,Pt2,EC);
[ 560137044461222 184453736165476 125 ]
[5] F=y^2-(x^3+EC[0]*x+EC[1])$
[6] subst(F,x,Pt3[0]/Pt3[2],y,Pt3[1]/Pt3[2]);
0
[7] ecm_add_ff(Pt3,ecm_chsgn_ff(Pt3),EC);
0
[8] D=ecm_sub_ff(Pt3,Pt2,EC);
[ 886545905133065 119584559149586 886545905133065 ]
[9] D[0]/D[2]==Pt1[0]/Pt1[2];
1
[10] D[1]/D[2]==Pt1[1]/Pt1[2];
1
```

References
          Section 10.5.1 [setmod_ff], page 178

Appendix A Appendix

A.1 Details of syntax

```
<expression>:
    '('<expression>')'
    <expression> <binary operator> <expression>
    '+' <expression>
    '-' <expression>
    <left value>
    <left value> <assignment operator> <expression>
    <left value> '++'
    <left value> '--'
    '++' <left value>
    '--' <left value>
    '!' <expression>
    <expression> '?' <expression> ':' <expression>
    <function> '(' <expr list> ')'
    <function> '(' <expr list> '|' <option list> ')'
    <string>
    <exponent vector>
    <atom>
    <list>
```
(See Section 4.2.10 [various expressions], page 25.)
```
<left value>:
    <program variable> ['['<expression>']']*
<binary operator>:
    '+' '-' '*' '/' '%' '^'(exponentiation)
    '==' '!=' '<' '>' '<=' '>=' '&&' '||'
    '==' '!=' '<' '>' '<=' '>=' '&&' '||'
<assignment operator>:
    '=' '+=' '-=' '*=' '/=' '%=' '^='
<expr list>:
    <empty>
    <expression> [',' <expression>]*
<option>:
    Character sequence beginning with an alphabetical letter '=' <expr>
<option list>:
    <option>
    <option> [',' <option>]*
<list>:
    '[' <expr list> ']'
<program variable>:
    Sequence of alphabetical letters or numeric digits or _
```

```
            that begins with a capital alphabetical letter
            (X,Y,Japan etc.)
```
(See Section 4.2.2 [variables and indeterminates], page 20.)

```
    <function>:
        Sequence of alphabetical letters or numeric digits or _
        that begins with a small alphabetical letter
        (fctr,gcd etc.)
    <atom>:
        <indeterminate>
        <number>
    <indeterminate>:
        Sequence of alphabetical letters or numeric digits or _
        that begin with a small alphabetical letter
        (a,bCD,c1_2 etc.)
```
(See Section 4.2.2 [variables and indeterminates], page 20.)

```
    <number>:
        <rational number>
        <floating point number>
        <algebraic number>
        <complex number>
```
(See Section 3.2 [Types of numbers], page 14.)

```
    <rational number>:
        0, 1, -2, 3/4
    <floating point number>:
        0.0, 1.2e10
    <algebraic number>:
        newalg(x^2+1), alg(0)^2+1
```
(See Chapter 9 [Algebraic numbers], page 162.)

```
    <complex number>:
        1+@i, 2.3*@i
    <string>:
        character sequence enclosed by two '"''s.
    <exponent vector>:
        '<<' <expr list> '>>'
```
(See Chapter 8 [Groebner basis computation], page 121.)

```
    <statement>:
        <expression> <terminator>
        <compound statement>
        'break' <terminator>
        'continue' <terminator>
        'return' <terminator>
        'return' <expression> <terminator>
        'if' '(' <expr list> ')' <statement>
```

```
                    'if' '(' <expr list> ')' <statement> 'else' <statement>
                    'for' '(' <expr list> ';' <expr list> ';' <expr list> ')' <statement>▮
                    'do' <statement> 'while' '(' <expr list> ')' <terminator>
                    'while' '(' <expr list> ')' <statement>
                    'def' <function> '(' <expr list> ')' '{' <variable declaration> <stat list> '}'▮
                    'end(quit)' <terminator>
```
(See Section 4.2.5 [statements], page 22.)

```
        <terminator>:
            ';' '$'

        <variable declaration>:
            ['extern' <program variable> [',' <program variable>]* <terminator>]*▮

        <compound statement>:
            '{' <stat list> '}'

        <stat list>:
            [<statement>]*
```

A.2 Files of user defined functions

There are several files of user defined functions under the standard library directory. ('/usr/local/lib/asir' by default.) Here, we explain some of them.

'fff'   Univariate factorizer over large finite fields (See Chapter 10 [Finite fields], page 176.)

'gr'   Groebner basis package.   (See Chapter 8 [Groebner basis computation], page 121.)

'sp'   Operations over algebraic numbers and factorization, Splitting fields.   (See Chapter 9 [Algebraic numbers], page 162.)

'alpi'
'bgk'
'cyclic'
'katsura'
'kimura'   Example polynomial sets for benchmarks of Groebner basis computation. (See Section 8.10.43 [katsura hkatsura cyclic hcyclic], page 158.)

'defs.h'   Macro definitions. (See Section 4.2.11 [preprocessor], page 26.)

'fctrtest'
             Test program of factorization of integral polynomials. It includes 'factor.tst' of REDUCE and several examples for large multiplicity factors. If this file is load()'ed, computation will begin immediately. You may use it as a first test whether **Asir** at you hand runs correctly.

'fctrdata'
             This contains example polynomials for factorization. It includes polynomials used in 'fctrtest'. Polynomials contained in vector Alg[] is for the algebraic factorization af(). (See Section 9.5.10 [asq af af_noalg], page 171.)

```
                    [45] load("sp")$
```

```
[84] load("fctrdata")$
[175] cputime(1)$
0msec
[176] Alg[5];
x^9-15*x^6-87*x^3-125
0msec
[177] af(Alg[5],[newalg(Alg[5])]);
[[1,1],[75*x^2+(10*#0^7-175*#0^4-470*#0)*x
+(3*#0^8-45*#0^5-261*#0^2),1],
[75*x^2+(-10*#0^7+175*#0^4+395*#0)*x
+(3*#0^8-45*#0^5-261*#0^2),1],
[25*x^2+(25*#0)*x+(#0^8-15*#0^5-87*#0^2),1],
[x^2+(#0)*x+(#0^2),1],[x+(-#0),1]]
3.600sec + gc : 1.040sec
```

`'ifplot'`   Examples for plotting. (See Section 7.5.15 [ifplot conplot plot polarplot plotover], page 116.) Vector `IS[]` contains several famous algebraic curves. Variables `H, D, C, S` contains something like the suits (Heart, Diamond, Club, and Spade) of cards.

`'num'`   Examples of simple operations on numbers.

`'mat'`   Examples of simple operations on matrices.

`'ratint'`   Indefinite integration of rational functions. For this, files 'sp' and 'gr' is necessary. A function `ratint()` is defined. Its returns a rather complex result.

```
[0] load("gr")$
[45] load("sp")$
[84] load("ratint")$
[102] ratint(x^6/(x^5+x+1),x);
[1/2*x^2,
[[(#2)*log(-140*x+(-2737*#2^2+552*#2-131)),
161*t#2^3-23*t#2^2+15*t#2-1],
[(#1)*log(-5*x+(-21*#1-4)),21*t#1^2+3*t#1+1]]]
```

In this example, indefinite integral of the rational function `x^6/(x^5+x+1)` is computed. The result is a list which comprises two elements: The first element is the rational part of the integral; The second part is the logarithmic part of the integral. The logarithmic part is again a list which comprises finite number of elements, each of which is of form `[root*log(poly),defpoly]`. This pair should be interpreted to sum up the expression `root*log(poly)` through all **root**'s root's of the `defpoly`. Here, `poly` contains `root`, and substitution for `root` is equally applied to `poly`. The logarithmic part in total is obtained by applying such interpretation to all element pairs in the second element of the result and then summing them up all.

`'primdec'`   Primary ideal decomposition of polynomial ideals and prime compotision of radicals over the rationals (see Section 8.10.44 [primadec primedec], page 159).

'`primdec_mod`'
>   Prime decomposition of radicals of polynomial ideals over finite fields (see
>   Section 8.10.45 [primedec_mod], page 159).

'`bfct`'   Computation of b-function. (see Section 8.10.46 [bfunction bfct generic_bfct
>   ann ann0], page 160).

A.3 Input interfaces

A command line editing facility and a history substitution facility are built-in for DOS,
Windows version of **Asir**. UNIX versions of **Asir** do not have such built-in facilities. Instead,
the following input interfaces are prepared. This are also available from our ftp server. As
for our ftp server See Section 1.3 [How to get Risa/Asir], page 2.

On Windows, '`asirgui.exe`' has a copy and paste functionality different from Windows
convention. Press the left button of the mouse and drag the mouse cursor on a text, then
the text is selected and is highlighted. When the button is released, highlighted text returns
to the normal state and it is saved in the copy buffer. If the right button is pressed, the
text in the copy buffer is inserted at the current text cursor position. Note that the existing
text is read-only and one cannot modify it.

A.3.1 fep

Fep is a general purpose front end processor. The author is K. Utashiro (SRA Inc.).

Under fep, emacs- or vi-like command line editing and csh-like history substitution are
available for UNIX commands, including '`asir`'.

```
% fep asir
...
[0] fctr(x^5-1);
[[1,1],[x-1,1],[x^4+x^3+x^2+x+1,1]]
[1] !!                                /* !!+Return                  */
fctr(x^5-1);                          /* The last input appears.    */
...                                   /* Edit+Return                */
fctr(x^5+1);
[[1,1],[x+1,1],[x^4-x^3+x^2-x+1,1]]
```

Fep is a free software and the source is available. However machines or operating systems on
which the original one can run are limited. The modified version by us running on several
unsupported environments is available from our ftp server.

A.3.2 asir.el

'`asir.el`' is a GNU Emacs interface for **Asir**.   The author is Koji Miyajima
(`YVE25250@pcvan.or.jp`). In '`asir.el`', completion of file names and command names is
realized other than the ordinary editing functions which are available on Emacs.

'`asir.el`' is distributed on PC-VAN. The version where several changes have been made
according to the current version of **Asir** is available via ftp.

The way of setting up and the usage can be found at the top of '`asir.el`'.

A.4 Library interfaces

It is possible to link an **Asir** library to use the functionalities of **Asir** from other programs. The necessary libraries are included in the **OpenXM** distribution (`http://www.math.kobe-u.ac.jp/OpenXM`). At present only the **OpenXM** interfaces are available. Here we assume that **OpenXM** is already installed. In the following `$OpenXM_HOME` denotes the **OpenXM** root directory. All the library files are placed in '`$OpenXM_HOME/lib`'. There are three kinds of libraries as follows.

- '`libasir.a`'
  It does not contain the functionalities related to **PARI** and **X11**. Only '`libasir-gc.a`' is necessary for linking.

- '`libasir_pari.a`'
  It does not contain the functionalities related to **X11**. '`libasir-gc.a`', '`libpari.a`' are necessary for linking.

- '`libasir_pari_X.a`'
  All the functionalities are included. '`libasir-gc.a`', '`libpari.a`' and libraries related to **X11** are necessary for linking.

- `int asir_ox_init(int `*`byteorder`*`)`
  It initializes the library. *byteorder* specifies the format of binary CMO data on the memory. If *byteorder* is 0, the byteorder native to the machine is used. If *byteorder* is 1, the network byteorder is used. It returns 0 if the initialization is successful, -1 otherwise.

- `void asir_ox_push_cmo(void *`*`cmo`*`)`

- `int asir_ox_peek_cmo_size()`
  It returns the size of the object at the top of the stack as CMO object. It returns -1 if the object cannot be converted into CMO object.

- `int asir_ox_pop_cmo(void *`*`cmo`*`, int `*`limit`*`)`
  It pops an **Asir** object at the top of the stack and it converts the object into CMO data. If the size of the CMO data is not greater than *limit*, then the data is written in *cmo* and the size is returned. Otherwise -1 is returned. The size of the array pointed by *cmo* must be at least *limit*. In order to know the size of converted CMO data in advance `asir_ox_peek_cmo_size` is called.

- `void asir_ox_push_cmd(int `*`cmd`*`)`
  It executes a stack machine command *cmd*.

- `void asir_ox_execute_string(char *`*`str`*`)`
  It evaluates *str* as a string written in the **Asir** user language. The result is pushed onto the stack.

A program calling the above functions should include '`$OpenXM_HOME/include/asir/ox.h`'. In this file all the definitions of **OpenXM** tags and commands. The following example ('`$OpenXM_HOME/doc/oxlib/test3.c`') illustrates the usage of the above functions.

```
#include <asir/ox.h>
#include <signal.h>

main(int argc, char **argv)
```

```
{
  char buf[BUFSIZ+1];
  int c;
  unsigned char sendbuf[BUFSIZ+10];
  unsigned char *result;
  unsigned char h[3];
  int len,i,j;
  static int result_len = 0;
  char *kwd,*bdy;
  unsigned int cmd;

  signal(SIGINT,SIG_IGN);
  asir_ox_init(1); /* 1: network byte order; 0: native byte order */
  result_len = BUFSIZ;
  result = (void *)malloc(BUFSIZ);
  while ( 1 ) {
    printf("Input>"); fflush(stdout);
    fgets(buf,BUFSIZ,stdin);
    for ( i = 0; buf[i] && isspace(buf[i]); i++ );
    if ( !buf[i] )
      continue;
    kwd = buf+i;
    for ( ; buf[i] && !isspace(buf[i]); i++ );
    buf[i] = 0;
    bdy = buf+i+1;
    if ( !strcmp(kwd,"asir") ) {
      sprintf(sendbuf,"%s;",bdy);
      asir_ox_execute_string(sendbuf);
    } else if ( !strcmp(kwd,"push") ) {
      h[0] = 0;
      h[2] = 0;
      j = 0;
      while ( 1 ) {
        for ( ; (c= *bdy) && isspace(c); bdy++ );
        if ( !c )
          break;
        else if ( h[0] ) {
          h[1] = c;
          sendbuf[j++] = strtoul(h,0,16);
          h[0] = 0;
        } else
          h[0] = c;
        bdy++;
      }
      if ( h[0] )
        fprintf(stderr,"Number of characters is odd.\n");
      else {
```

```
            sendbuf[j] = 0;
            asir_ox_push_cmo(sendbuf);
          }
        } else if ( !strcmp(kwd,"cmd") ) {
          cmd = atoi(bdy);
          asir_ox_push_cmd(cmd);
        } else if ( !strcmp(kwd,"pop") ) {
          len = asir_ox_peek_cmo_size();
          if ( !len )
            continue;
          if ( len > result_len ) {
            result = (char *)realloc(result,len);
            result_len = len;
          }
          asir_ox_pop_cmo(result,len);
          printf("Output>"); fflush(stdout);
          printf("\n");
          for ( i = 0; i < len; ) {
            printf("%02x ",result[i]);
            i++;
            if ( !(i%16) )
              printf("\n");
          }
          printf("\n");
        }
      }
    }
```

This program receives a line in the form of *keyword body* as an input and it executes the following operations according to *keyword*.

- `asir` *body*
  body is regarded as an expression written in the **Asir** user language. The expression is evaluated and the result is pushed onto the stack. `asir_ox_execute_string()` is called.

- `push` *body*
  body is regarded as a CMO object in the hexadecimal form. The CMO object is converted into an **Asir** object and is pushed onto the stack. `asir_ox_push_cmo()` is called.

- `pop`
  The object at the top of the stack is converted into a CMO object and it is displayed in the hexadecimal form. `asir_ox_peek_cmo_size()` and `asir_ox_pop_cmo()` are called.

- `cmd` *body*
  body is regarded as an SM command and the command is executed. `asir_ox_push_cmd()` is called.

A.5 Appendix

A.5.1 ChangeLog

See the Japanese document and the cvsweb at `http://www.openxm.org`

A.5.2 Version 990831

Four years have passed since the last distribution. Though the look and feel seem unchanged, internally there are several changes such as 32-bit representation of bignums. Plotting facilities are not available on Windows.

If you have files created by `bsave` on the older version, you have to use `bload27` to read such files.

A.5.3 Version 950831

A.5.3.1 Debugger

- One can enter the debug mode anytime.
- A command `finish` has been appended.
- One can examine any stack frame with `up`, `down` and `frame`.
- A command `trace` has been appended.

A.5.3.2 Built-in functions

- One can specify a main variable for `sdiv()` etc.
- Functions for polynomial division over finite fields such as `sdivm()` have been appended.
- `det()`, `res()` can produce results over finite fields.
- `vtol()`, conversion from a vector to a list has been appended.
- `map()` has been appended.

A.5.3.3 Groebner basis computation

- Functions for Groebner basis computation have been implemented as built-in functions.
- `grm()` and `hgrm()` have been changed to `gr()` and `hgr()` respectively.
- `gr()` and `hgr()` requires explicit specification of an ordering type.
- Extension of specification of a term ordering type.
- Groebner basis computations over finite fields.
- Lex order Groebner basis computation via a modular change of ordering algorithm.
- Several new built-in functions.

A.5.3.4 Others

- Implementation of tools for distributed computation.
- Application of modular computation for GCD computation over algebraic number fields.
- Implementation of primary decomposition of ideals.
- Porting to Windows.

A.5.4 Version 940420

The first public version.

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