Quadratic relations for confluent hypergeometric functions

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Abstract

We present a theory of intersection on the complex projective line for homology and cohomology groups defined by connections which are regular or not. We apply this theory to confluent hypergeometric functions, and obtain, as an analogue of period relations, quadratic relations satisfied by confluent hypergeometric functions.

1 Introduction

The main objective of this paper is to provide a systematic method of deriving new quadratic relations for confluent hypergeometric functions, especially, in several variables. Classical examples of the quadratic relations are the inversion formula for the gamma function

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

and Lommel's formula for Bessel functions

$$J_a(z)J_{-a+1}(z) + J_{a-1}(z)J_{-a}(z) = \frac{2\sin(\pi a)}{\pi z}.$$

The essence of our method is to regard these quadratic relations as analogs of Riemann's period relations, which are quadratic relations for periods on a compact Riemann surface. Periods are integrals of holomorphic 1-forms (1-cocycles)

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over closed paths (1-cycles) on the Riemann surface. The naturality of the pairings of the cohomology and homology groups of the Riemann surface yields period relations. The coefficients of the period relation can be understood as intersection numbers of the cycles and the cocycles.

We regard integral representations of confluent hypergeometric functions as pairings of cocycles of certain of cohomology group and cycles of certain of homology group. We will introduce the intersection pairing between the cohomology group and its dual, which naturally induces the intersection pairing between the homology group and its dual. The naturality of the pairings yields quadratic relations for confluent hypergeometric functions, as in the case of Riemann's period relations.

We note that the existence of the quadratic relations is an immediate consequence of the commutativity of the dualizing functor and the integration functor, which yields the cohomology groups, for $\mathcal{D}$-modules. However, the authors would like to emphasize that we are interested in deriving *explicit* formulas for hypergeometric functions, and that such a general fact is not satisfactory for us.

Let us explain what the cohomology and homology groups are and where the difficulty lies. Let ω be a rational 1-form on the complex projective line $\mathbf{P}$ with the polar set $x = \{x_1, \ldots, x_m\}$ such that the residue at any simple pole is not an integer. Let $\mathcal{L}_\omega$ and $\mathcal{L}_{-\omega}$ be the locally constant sheaves over $X = \mathbf{P} \setminus x$ of analytic functions $u(t)$ and $u^{-1}(t)$ satisfying $\nabla_{-\omega} u(t) = 0$ and $\nabla_\omega u^{-1}(t) = 0$, respectively, where $\nabla_\omega = d + \omega\wedge$ and $\nabla_{-\omega} = d - \omega\wedge$. Note that such $u(t)$ is expressed as $c\exp(\int^t \omega)$ $(c \in \mathbf{C})$. We consider the twisted cohomology groups $H^1(\Omega^\bullet(x), \nabla_{\pm\omega}) = \Omega^1(x)/\nabla_{\pm\omega}(\Omega^0(x))$, where $\Omega^k(x)$ denotes the vector space of rational k-forms admitting poles in x, and the twisted homology groups $H_1(X, \mathcal{L}_{\pm\omega})$. When the 1-form ω admits only simple poles, the intersection pairing for $H^1(\Omega^\bullet(x), \nabla_{\pm\omega})$ and that for the twisted homology groups $H_1(X, \mathcal{L}_{\pm\omega})$

are studied. Following de Rham's original work in [3], Kita and Yoshida gave evaluation formulas for intersection numbers of homology in [11]. Subsequently, evaluation formulas for intersection numbers for cohomology were established and some quadratic relations for Lauricella's F_D's were given in [1]. It is fundamental in these papers that $H^1(\Omega^\bullet(x), \nabla_\omega)$ is isomorphic to

$$H^1(E_c^\bullet(x), \nabla_\omega) = \frac{\ker(\nabla_\omega : E_c^1(x) \to E_c^2(x))}{\nabla_\omega E_c^0(x)},$$

where $E_c^k(x)$ denotes the space of smooth k-forms on X with compact support, and that both of $H^1(\Omega^\bullet(x), \nabla_{-\omega})$ and $H_1(X, \mathcal{L}_\omega)$ can be regarded as the dual space of $H^1(E_c^\bullet(x), \nabla_\omega)$. For a rational 1-form ω with higher order poles, the groups $H^1(E_c^\bullet(x), \nabla_\omega)$ and $H_1(X, \mathcal{L}_\omega)$ are well-defined, but $H^1(\Omega^\bullet(x), \nabla_\omega)$ is not isomorphic to $H^1(E_c^\bullet(x), \nabla_\omega)$ in general and $H_1(X, \mathcal{L}_\omega)$ is too small to form a fundamental system of solutions for a confluent hypergeometric system of differential equations. In order to generalize results in [1] and [11], we need to find suitable cohomology and homology groups to express confluent hypergeometric functions.

To this end, we modify the isomorphic theorem for an integrable connection provided by the first author in [13] by replacing the asymptotic parts by C^∞ object. The key role is played by the isomorphism

$$\iota_\omega : H^1(\Omega^\bullet(x), \nabla_\omega) \to H^1(S^\bullet(x), \nabla_\omega),$$

where $S^\bullet(x)$ is the complex of the space of rapidly decreasing k-forms on X (see Section 2). This isomorphism induces the intersection pairing between $H^1(\Omega^\bullet(x), \nabla_\omega)$ and $H^1(\Omega^\bullet(x), \nabla_{-\omega})$ by

$$\int_X \iota_\omega(\varphi^+) \wedge \varphi^-.$$

In order to evaluate intersection numbers, we give an explicit form for the image $\varphi \in \Omega^1(x)$ under the isomorphism ι_ω.

We introduce a homology group $H_1(C^{\omega}_{\bullet}(X),\partial_{\omega})$ so that the pairings between an element φ of $H^1(\Omega^{\bullet}(x),\nabla_{\omega})$ and a base of $H_1(C^{\omega}_{\bullet}(X),\partial_{\omega})$ form a fundamental system of solutions for a confluent hypergeometric system of differential equations (see Section 3). We show the perfectness of the pairing between $H^1(S^{\bullet}(x),\nabla_{\omega})$ and $H_1(C^{\omega}_{\bullet}(X),\partial_{\omega})$. This together with the perfect pairing between $H^1(\Omega^{\bullet}(x),\nabla_{\pm\omega})$ shown in [7] induce the perfect pairing between $H_1(C^{\omega}_{\bullet}(X),\partial_{\omega})$ and $H_1(C^{-\omega}_{\bullet}(X),\partial_{-\omega})$. We present a formula to evaluate intersection numbers between $H_1(C^{\omega}_{\bullet}(X),\partial_{\omega})$ and $H_1(C^{-\omega}_{\bullet}(X),\partial_{-\omega})$ by comparison theorems given by Malgrange in [14]. We give an explicit intersection matrix I_{h} for certain elements of $H_1(C^{\pm\omega}_{\bullet}(X),\partial_{\pm\omega})$ by this formula.

We have begun with the wedge product of globally defined differential forms to discuss our method of deriving quadratic relations. An anonymous referee advised us that we should start from the Poincaré-Verdier duality between the locally constant sheaves $\mathcal{L}_{\omega}$ and $\mathcal{L}_{-\omega}$ on the real blowing up space of $\mathbf{P}$ with the centers in x (see, e.g., [9, Chapter 3]), which induces duality of cohomology groups. Although we agree that it is a modern and attractive approach, we do not think that it will drastically simplify our discussions, because we need explicit constructions of isomorphisms between cohomology groups and the Poincaré duality.

Different approaches to derive quadratic relations for confluent hypergeometric functions are given by Sasaki and Yoshida in [21] and by Haraoka in [6].

2 Twisted de Rham cohomology groups

Let $n_1,\dots,n_m$ be natural numbers satisfying

$$n_1 \ge n_2 \ge \dots \ge n_m, \quad n=\sum_{i=1}^{m} n_i \ge 3$$

and let $x_1, \ldots, x_m$ be m distinct points on the complex projective line $\mathbf{P}$. Put

$$\sigma = \#\{i \mid n_i > 1\}, \quad x = \{x_1, \ldots, x_m\}, \quad X = \mathbf{P} \setminus x$$

and let

$$\omega = \sum_{i=1}^{m} \left(\frac{\alpha_{i;1}}{t - x_i} + \frac{\alpha_{i;2}}{(t - x_i)^2} + \cdots + \frac{\alpha_{i;n_i-1}}{(t - x_i)^{n_i-1}} + \frac{\alpha_{i;n_i}}{(t - x_i)^{n_i}} \right) dt$$

be a rational 1-form, where $\alpha_{i;k} \in \mathbf{C}$, $\alpha_{i;n_i} \neq 0$ for all i, $\alpha_{i;1} \notin \mathbf{Z}$ in case $n_i = 1$, and

$$\sum_{i=1}^{m} \alpha_{i;1} = 0.$$

Throughout this paper, we assume this condition on the parameters $\alpha_{i;k}$ *,* is satisfied. For the 1-form ω on $\mathbf{P}$, we denote by $\nabla_\omega = d + \omega\wedge$ the connection with respect to ω on X; note that $\nabla_\omega \circ \nabla_\omega = 0$.

A smooth function f defined in a neighborhood U_i of x_i is said to be rapidly decreasing at x_i if f satisfies

$$\lim_{t \to x_i} \frac{1}{|t - x_i|^r} \frac{\partial^{p+q}}{\partial t^p \partial \bar{t}^q} f(t) = 0$$

for any $p, q, r \in \{0, 1, 2, \ldots\}$, where t is a complex coordinate system around x_i. Let $S^0(x)$ be the vector space of smooth functions on $\mathbf{P}$ which rapidly decrease at x_i for any i, and $S^k(x)$ the vector space of smooth k-forms ζ on $\mathbf{P}$ such that the coefficients in the expression of ζ in terms of a complex coordinate system t around x_i rapidly decrease at x_i for any i. We denote the sheaf over $\mathbf{P}$ of such k-forms by $\mathcal{S}^k(x)$.

A smooth function f defined on $U_i \setminus \{x_i\}$ is said to grow t-polynomially at x_i if there exists $r \in \mathbf{N}$ such that $(t - x_i)^r f(t)$ is smooth on U_i. Let $P^0(x)$ be the vector space of smooth functions f on X which grow t-polynomially at x_i for any i, and $P^k(x)$ the vector space of smooth k-forms ζ such that the coefficients in the expression of ζ in terms of a complex coordinate system t around x_i grow t-polynomially at x_i for any i. We denote the sheaf over $\mathbf{P}$ of such k-forms and that of such (p,q)-forms by $\mathcal{P}^k(x)$ and $\mathcal{P}^{(p,q)}(x)$, respectively. Note that $\Gamma(X,\mathcal{P}^{(p,q)}) = P^{(p,q)}$ and that the stalk $\mathcal{P}^{(p,q)}_{x_i}$ of $\mathcal{P}^{(p,q)}$ at x_i is equal to $\mathcal{E}^{(p,q)}_{x_i}[1/(t-x_i)]$, where $\mathcal{E}^{(p,q)}_{x_i}$ is the stalk of the sheaf $\mathcal{E}^{(p,q)}$ of smooth (p,q)-forms over $\mathbf{P}$ at x_i.

We define three complexes with differential ∇_ω :

$$\begin{aligned}
(\Omega^\bullet(x),\nabla_\omega) &: \quad \Omega^0(x)\xrightarrow{\nabla_\omega}\Omega^1(x)\xrightarrow{\nabla_\omega} 0 \longrightarrow 0,\\
(S^\bullet(x),\nabla_\omega) &: \quad S^0(x)\xrightarrow{\nabla_\omega}S^1(x)\xrightarrow{\nabla_\omega}S^2(x)\xrightarrow{\nabla_\omega} 0,\\
(P^\bullet(x),\nabla_\omega) &: \quad P^0(x)\xrightarrow{\nabla_\omega}P^1(x)\xrightarrow{\nabla_\omega}P^2(x)\xrightarrow{\nabla_\omega} 0,
\end{aligned}$$

where $\Omega^k(x)$ is the vector space of rational k-forms on $\mathbf{P}$ admitting poles at x. The k-th cohomology groups $H^k(\Omega^\bullet(x),\nabla_\omega)$, $H^k(S^\bullet(x),\nabla_\omega)$ and $H^k(P^\bullet(x),\nabla_\omega)$ of the above complexes are called rational, rapidly decreasing and t-polynomially growing twisted de Rham cohomology groups with respect to ∇_ω, respectively. The inclusions

$$(\Omega^\bullet(x),\nabla_\omega)\subset(P^\bullet(x),\nabla_\omega),\quad (S^\bullet(x),\nabla_\omega)\subset(P^\bullet(x),\nabla_\omega)$$

of complexes induce the following isomorphisms among twisted de Rham cohomology groups.

Theorem **2.1** (cf. [12, Theorem 2], [13, Proposition 3.1], [14, p.82 ii)]) *We have*

$$H^k(\Omega^\bullet(x),\nabla_\omega)\simeq H^k(P^\bullet(x),\nabla_\omega)\simeq H^k(S^\bullet(x),\nabla_\omega).$$

For $k\neq 1$, $H^k(\Omega^\bullet(x),\nabla_\omega)$, $H^k(P^\bullet(x),\nabla_\omega)$ and $H^k(S^\bullet(x),\nabla_\omega)$ vanish.

It is shown in [10] that the dimension of $H^1(\Omega^\bullet(x),\nabla_\omega)$ is $n-2$, which is equal to the rank of the associated confluent hypergeometric system of differential equations. See also [20].

Remark 2.1 Let $E^k(x)$ be the space of smooth k-forms on X, and $E^k_c(x)$ the space of smooth k-forms with compact support on X. When the 1-form ω admits only simple poles with non-integral residue, we have the isomorphisms (cf. [2, Corollary 6.3])

$$H^1(\Omega^\bullet(x), \nabla_\omega) \simeq H^1(E^\bullet(x), \nabla_\omega) \simeq H^1(E_c^\bullet(x), \nabla_\omega),$$

which were fundamental for the study of intersection numbers in [11]. On the other hand, we have

$$H^1(E^\bullet(x), \nabla_\omega) \not\simeq H^1(\Omega^\bullet(x), \nabla_\omega), \quad H^1(E_c^\bullet(x), \nabla_\omega) \not\simeq H^1(\Omega^\bullet(x), \nabla_\omega)$$

for a rational ω with higher order poles. This is the reason why we introduce rapidly decreasing and t-polynomially growing twisted de Rham cohomology groups.

The first author proved an isomorphism theorem in [13], which yields essentially the following isomorphism

$$H^1(\Omega^\bullet(x), \nabla_\omega) \simeq H^1(S^\bullet(x), \nabla_\omega),$$

for a rational 1-form ω with non-integral residues at simple poles. In order to derive explicit formulas for intersection numbers, we will give an elementary proof of Theorem 2.1 in the rest of this section.

We start with proving the following lemma on the $\bar\partial$ equation. Let $\mathit{\Omega}^p(x)$ be the sheaf of meromorphic p-forms over $\mathbf{P}$ admitting poles on x.

LEMMA **2.2** (1).
The sequence of sheaves

$$0 \longrightarrow \mathit{\Omega}^p(x) \xrightarrow{\mathrm{id}} \mathcal{P}^{(p,0)}(x) \xrightarrow{\bar\partial} \mathcal{P}^{(p,1)}(x) \longrightarrow 0$$

is exact for $p = 0, 1$. (2). *The sequence*

$$0 \longrightarrow \Gamma(\mathbf{P}, \mathit{\Omega}^p(x)) \xrightarrow{\mathrm{id}} \Gamma(\mathbf{P}, \mathcal{P}^{(p,0)}(x)) \xrightarrow{\bar\partial} \Gamma(\mathbf{P}, \mathcal{P}^{(p,1)}(x)) \longrightarrow 0$$

is exact for $p = 0, 1$.

Proof. It is well-known that $\bar\partial$ is surjective on the germ of smooth $(p,1)$-forms (see, e.g., [5, p.25]). Let $U \ni x_i$ be an open set and suppose that we are given $g \in \mathcal{P}^{(p,1)}(U)$. By definition, there exists a number r such that $(t-x_i)^r g$ is a smooth function. From the surjectivity for smooth $(p,1)$-forms, there exists a smooth $(p,0)$-form f such that $\bar\partial f = (t-x_i)^r g$. Since $(t-x_i)$ commutes with $\bar\partial$, we have $\bar\partial(f/(t-x_i)^r) = g$. Hence, $\bar\partial$ is surjective. It is clear that the kernel of $\bar\partial : \mathcal{P}^{(p,0)} \to \mathcal{P}^{(p,1)}$ is the germ of meromorphic functions with poles at x.

The second statement follows from the well-known vanishing of $H^1(\mathbf{P}, \Omega^p(x))$ (see, e.g., [4, p.141, 17.17]) and a long exact sequence. q.e.d.

Proof of

$$H^k(\Omega^\bullet(x), \nabla_\omega) \simeq H^k(P^\bullet(x), \nabla_\omega).$$

Let us consider the double complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^0(x) & \xrightarrow{\mathrm{id}} & \mathcal{P}^{(0,0)}(x) & \xrightarrow{\bar\partial} & \mathcal{P}^{(0,1)}(x) & \longrightarrow & 0 \\
 & & \downarrow \nabla_\omega & & \downarrow \partial+\omega & & \downarrow \partial+\omega & & \\
0 & \longrightarrow & \Omega^1(x) & \xrightarrow{\mathrm{id}} & \mathcal{P}^{(1,0)}(x) & \xrightarrow{\bar\partial} & \mathcal{P}^{(1,1)}(x) & \longrightarrow & 0,
\end{array}$$

where each row is exact and $\nabla_\omega = d+\omega = \partial+\bar\partial+\omega$. Since $H^k(\mathbf{P}, \Omega^p(x))$ vanishes for $k \geq 1$, we obtain the following double complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\mathbf{P}, \Omega^0(x)) & \xrightarrow{\mathrm{id}} & \Gamma(\mathbf{P}, \mathcal{P}^{(0,0)}(x)) & \xrightarrow{\bar\partial} & \Gamma(\mathbf{P}, \mathcal{P}^{(0,1)}(x)) & \longrightarrow & 0 \\
 & & \downarrow \nabla_\omega & & \downarrow \partial+\omega & & \downarrow \partial+\omega & & \\
0 & \longrightarrow & \Gamma(\mathbf{P}, \Omega^1(x)) & \xrightarrow{\mathrm{id}} & \Gamma(\mathbf{P}, \mathcal{P}^{(1,0)}(x)) & \xrightarrow{\bar\partial} & \Gamma(\mathbf{P}, \mathcal{P}^{(1,1)}(x)) & \longrightarrow & 0.
\end{array}$$

By a standard argument in homological algebra, $H^i(\Gamma(\mathbf{P}, \Omega^\bullet(x)), \nabla_\omega)$ is equal to the cohomology of the associated single complex of the double complex

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma(\mathbf{P}, \mathcal{P}^{(0,0)}(x)) & \xrightarrow{\bar\partial} & \Gamma(\mathbf{P}, \mathcal{P}^{(0,1)}(x)) & \longrightarrow & 0 \\
 & & \downarrow \partial+\omega & & \downarrow \partial+\omega & & \\
0 & \longrightarrow & \Gamma(\mathbf{P}, \mathcal{P}^{(1,0)}(x)) & \xrightarrow{\bar\partial} & \Gamma(\mathbf{P}, \mathcal{P}^{(1,1)}(x)) & \longrightarrow & 0.
\end{array}$$

q.e.d

We next consider the ring $\mathbf{C}_{x_i}[[t,\bar t]]$ of formal power series around x_i. Put

$$\mathcal{F}^0_{x_i} = \mathbf{C}_{x_i}[[t,\bar t]], \quad \mathcal{F}^1_{x_i} = \mathcal{F}^0_{x_i}dt \oplus \mathcal{F}^0_{x_i}d\bar t, \quad \mathcal{F}^2_{x_i} = \mathcal{F}^0_{x_i}dt \wedge d\bar t,$$

$$\tilde{\mathcal{F}}^0_{x_i} = \mathbf{C}_{x_i}[[t,\bar{t}]][\frac{1}{t-x_i}], \quad \tilde{\mathcal{F}}^1_{x_i} = \tilde{\mathcal{F}}^0_{x_i}dt \oplus \tilde{\mathcal{F}}^0_{x_i}d\bar{t}, \quad \tilde{\mathcal{F}}^2_{x_i} = \tilde{\mathcal{F}}^0_{x_i}dt \wedge d\bar{t}.$$

It is known that the sequence

$$0 \longrightarrow \mathcal{S}^k_{x_i}(x) \longrightarrow \mathcal{E}^k_{x_i} \longrightarrow \mathcal{F}^k_{x_i} \longrightarrow 0$$

is exact, where $\mathcal{E}^k_{x_i}$ is the stalk at x_i of the sheaf $\mathcal{E}^k$ of smooth k-forms over $\mathbf{P}$.

Lemma **2.3** *The k-th cohomology group $H^k(\tilde{\mathcal{F}}^\bullet_{x_i}, \nabla_\omega)$ of the complex*

$$(\tilde{\mathcal{F}}^\bullet_{x_i}, \nabla_\omega) \ : \ \tilde{\mathcal{F}}^0_{x_i} \xrightarrow{\nabla_\omega} \tilde{\mathcal{F}}^1_{x_i} \xrightarrow{\nabla_\omega} \tilde{\mathcal{F}}^2_{x_i} \xrightarrow{\nabla_\omega} 0$$

vanishes for $k = 0, 1, 2$.

Proof. Suppose that there exists a non-zero $f \in \tilde{\mathcal{F}}^0_{x_i}$ such that $\nabla_\omega f = 0$. Since $\bar{\partial} f = 0$, f consists of terms $c_\nu(t-x_i)^\nu$. Let N be the minimum number such that $c_N \neq 0$. The leading term of $(\partial + \omega\wedge)f$ is

$$(\delta(n_i, 1)N + \alpha_{i;n_i})c_N(t-x_i)^{N-n_i}dt,$$

which does not vanish by our assumption on the parameters $\alpha_{i;n_i}$ and $\alpha_{i;1}$. This contradicts $\nabla_\omega f = 0$, which implies $H^0(\tilde{\mathcal{F}}^\bullet_{x_i}(x), \nabla_\omega) = 0$.

Suppose that $f = f_1 dt + f_2 d\bar{t} \in \tilde{\mathcal{F}}^1_{x_i}$ satisfies $\nabla_\omega f = 0$. Since f_2 does not contain terms $(\bar{t} - \bar{x}_i)^{-\mu}$ ($\mu \in \mathbf{N}$), there exists $F \in \tilde{\mathcal{F}}^0_{x_i}$ such that $\bar{\partial} F = f_2 d\bar{t}$. The element

$$f - \nabla_\omega F = f_1 dt - (\partial + \omega\wedge)F + f_2 d\bar{t} - \bar{\partial} F =: g dt \in \tilde{\mathcal{F}}^0_{x_i}dt,$$

satisfies

$$\bar{\partial}(g dt) = \nabla_\omega g dt = \nabla_\omega(f - \nabla_\omega F) = \nabla_\omega f - \nabla_\omega \circ \nabla_\omega F = 0,$$

which implies that g consists of terms $c_\nu(t-x_i)^\nu$. Express ω as an element of $\tilde{\mathcal{F}}^0_{x_i}dt$, put

$$G = \sum_{\nu=N}^{\infty} b_\nu(t-x_i)^\nu \quad (N \in \mathbf{Z})$$

and write down the equation $\nabla_\omega G = gdt$. We can easily find that there exist a unique G such that $\nabla_\omega G = gdt$ by our assumption on the parameters $\alpha_{i;n_i}$ and $\alpha_{i;1}$. Hence we have $\nabla_\omega(F+G) = f$, which implies $H^1(\tilde{\mathcal{F}}^\bullet_{x_i}(x), \nabla_\omega) = 0$.

For any $f \in \tilde{\mathcal{F}}^2_{x_i}$, we have already seen that there exists $Fdt \in \tilde{\mathcal{F}}^0_{x_i}dt$ such that $\bar{\partial}Fdt = \nabla_\omega F dt = f$, which implies $H^2(\tilde{\mathcal{F}}^\bullet_{x_i}(x), \nabla_\omega) = 0$. q.e.d.

Proof of

$$H^k(S^\bullet(x), \nabla_\omega) \simeq H^k(P^\bullet(x), \nabla_\omega).$$

Since the sequence

$$0 \longrightarrow \mathcal{S}^k_{x_i}(x) \longrightarrow \mathcal{P}^k_{x_i}(x) \longrightarrow \tilde{\mathcal{F}}^k_{x_i} \longrightarrow 0$$

is exact and $\mathcal{S}^k(x)$ is a fine sheaf, we have the following exact sequence of complexes

$$0 \longrightarrow (S^\bullet(x), \nabla_\omega) \longrightarrow (P^\bullet(x), \nabla_\omega) \longrightarrow \bigoplus_{i=1}^m (\tilde{\mathcal{F}}^\bullet_{x_i}, \nabla_\omega) \longrightarrow 0. \qquad (1)$$

The previous lemma shows that the k-th cohomology groups of the complexes $(S^\bullet(x), \nabla_\omega)$ and $(P^\bullet(x), \nabla_\omega)$ are isomorphic. q.e.d.

Let us now explicitly construct a cocycle in $H^1(S^\bullet(x), \nabla_\omega)$ corresponding to φ of $H^1(\Omega^\bullet(x), \nabla_\omega)$ under the isomorphism

$$\iota_\omega : H^1(\Omega^\bullet(x), \nabla_\omega) \to H^1(S^\bullet(x), \nabla_\omega).$$

By Lemma 2.3, for each x_i, there exists

$$G_i = G_i^1 + G_i^2 = \sum_{\nu=-N}^{N} c_\nu (t-x_i)^\nu + \sum_{\nu=N+1}^{\infty} c_\nu (t-x_i)^\nu \in \tilde{\mathcal{F}}^0_{x_i} \qquad (2)$$

such that

$$\nabla_\omega G_i = \varphi \in \Omega^1_{x_i}(x) \subset \tilde{\mathcal{F}}^1_{x_i},$$

where N is a sufficiently large integer. The exact sequence

$$0 \longrightarrow \mathcal{S}^0_{x_i}(x) \longrightarrow \mathcal{E}^0_{x_i} \longrightarrow \mathcal{F}^0_{x_i} \longrightarrow 0$$

implies that there exists a smooth function F_i around x_i such that the formal Taylor series of F_i at x_i is equal to G_i^2. We have

$$f_i = G_i^1 + F_i \in \mathcal{P}^0_{x_i}(x), \quad \varphi - \nabla_\omega f_i \in \mathcal{S}^1_{x_i}(x). \tag{3}$$

Though each f_i is defined only in a small neighborhood U_i of x_i, we can regard $h_i \cdot f_i$ to be defined on X, where h_i is a smooth function on $\mathbf{P}$ such that

$$\begin{aligned} h_i(t) = 1, &\qquad t \in V_i, \\ 0 \le h_i(t) \le 1, &\qquad t \in U_i \setminus V_i, \\ h_i(t) = 0, &\qquad t \notin U_i, \end{aligned}$$

for $x_i \in V_i \subset U_i$. The element

$$\iota_\omega(\varphi) = \varphi - \sum_{i=1}^m \nabla_\omega(h_i \cdot f_i) = \varphi - \sum_{i=1}^m (h_i \cdot \nabla_\omega(f_i) + f_i \cdot dh_i) \tag{4}$$

belongs to $\ker(\nabla_\omega : S^1(x) \to S^2(x))$ and is cohomologous to φ in $H^1(P^\bullet(x), \nabla_\omega)$.

We close this section with the following proposition which is necessary to define intersection numbers later.

Proposition **2.4** *The k-th cohomology group $H^k(S^\bullet(x), d)$ of the complex*

$$(S^\bullet(x), d) \ : \ S^0(x) \xrightarrow{d} S^1(x) \xrightarrow{d} S^2(x) \xrightarrow{d} 0$$

is isomorphic to the k-th de Rham cohomology group $H^k_{DR}(\mathbf{P}, \mathbf{C})$ of $\mathbf{P}$. In particular, we have

$$H^2(S^\bullet(x), d) \simeq \mathbf{C};$$

the isomorphism is given by

$$S^2(x) \ni \varphi \mapsto \int_{\mathbf{P}} \varphi \in \mathbf{C}.$$

Proof. It is known that the sequence

$$(\mathcal{F}^{\bullet}_{x_i}, d) \ : \ 0 \longrightarrow \mathcal{F}^0_{x_i} \xrightarrow{d} \mathcal{F}^1_{x_i} \xrightarrow{d} \mathcal{F}^2_{x_i} \xrightarrow{d} 0$$

is exact. The exact sequence

$$0 \longrightarrow \mathcal{S}^k_{x_i}(x) \longrightarrow \mathcal{E}^k_{x_i} \longrightarrow \mathcal{F}^k_{x_i} \longrightarrow 0$$

yields the following exact sequence of complexes

$$0 \longrightarrow (S^{\bullet}(x), d) \longrightarrow (E^{\bullet}, d) \longrightarrow \bigoplus_{i=1}^{m} (\mathcal{F}^{\bullet}_{x_i}, d) \longrightarrow 0,$$

where $(E^{\bullet}, d)$ is the de Rham complex on $\mathbf{P}$. Then $H^k(S^{\bullet}(x), d)$ is isomorphic to $H^k_{DR}(\mathbf{P}, \mathbf{C})$. Note that the isomorphism from $H^2(S^{\bullet}(x), d)$ to $H^2_{DR}(\mathbf{P}, \mathbf{C})$ is given by the natural inclusion and the map from $H^2_{DR}(\mathbf{P}, \mathbf{C})$ to $\mathbf{C}$ is by $\varphi \mapsto \int_{\mathbf{P}} \varphi$.

q.e.d.

3 Twisted homology groups

Let $\mathcal{L}_\omega$ be the locally constant sheaf over X of analytic functions which belong to the kernel of the connection $\nabla_{-\omega}$. Since a solution of the differential equation $\nabla_{-\omega} f(t) = 0$ can be locally expressed as $c \exp\left(\int_s^t \omega\right)$ $(c \in \mathbf{C})$ around any point s in X, $\mathcal{L}_\omega$ is determined by the multi-valued function

$$u(t) = \prod_{i=1}^{m} (t - x_i)^{\alpha_{i;1}} \exp\left(-\frac{\alpha_{i;2}}{(t - x_i)} - \frac{\alpha_{i;3}}{2(t - x_i)^2} - \cdots - \frac{\alpha_{i;n_i}}{(n_i - 1)(t - x_i)^{n_i - 1}}\right)$$

on X. Let $C^\omega_k(X)$ be the vector space of finite sums of formal products $\rho_i \otimes u_{\rho_i}(t)$, where ρ_i is a smooth k-chain in $\mathbf{P} \setminus \{x_i \,|\, n_i = 1\}$ and $u_{\rho_i}(t)$ is a branch of $u(t)$ on $\rho_i \cap X$ such that *$u_{\rho_i}(t)$ can be continuously extended to 0 at every point of $\rho_i \cap x$* if the set $\rho_i \cap x$ is not empty. We define a boundary operator ∂_ω on $C^\omega_{\bullet}(X)$ as $\partial_\omega : \rho \otimes u_\rho(t) \mapsto \partial\rho \otimes u_\rho(t)|_{\partial\rho}$, where ∂ is the ordinary boundary operator and

$u_\rho(t)|_{\partial\rho}$ is the restriction of $u_\rho(t)$ on $\partial\rho$. Since $\partial_\omega \circ \partial_\omega = 0$, we have the complex with boundary operator ∂_ω

$$(C^\omega_\bullet(X), \partial_\omega) : C^\omega_2(X) \xrightarrow{\partial_\omega} C^\omega_1(X) \xrightarrow{\partial_\omega} C^\omega_0(X) \longrightarrow 0.$$

The k-th homology group of this complex is denoted by $H_k(C^\omega_\bullet(X), \partial_\omega)$.

We define a pairing between $S^k(x)$ and $C^\omega_k(X)$ by

$$\langle \varphi, \gamma \rangle = \sum_\nu b_\nu \int_{\rho_\nu} u_{\rho_\nu}(t)\varphi,$$

where

$$\varphi \in S^k(x), \ \gamma = \sum_\nu b_\nu \rho_\nu \otimes u_{\rho_\nu}(t) \in C^\omega_k(X).$$

Since we have $\langle \varphi + \nabla_\omega f, \ \gamma + \partial_\omega g \rangle = \langle \varphi, \gamma \rangle$ for $\varphi \in \ker(\nabla_\omega : S^k(x) \to S^{k+1}(x))$, $f \in S^{k-1}(x)$, $\gamma \in \ker(\partial_\omega : C^\omega_k(X) \to C^\omega_{k-1}(X))$ and $g \in C^\omega_{k+1}(X)$ by the Stokes theorem, this pairing descends to the pairing of $H^k(S^\bullet(x), \nabla_\omega)$ and $H_k(C^\omega_\bullet(X), \partial_\omega)$.

Let us now introduce some elements of $C^\omega_k(X)$. Fix x and α, take $x_0 \in X$ and $c \in \mathbf{C} \setminus \{0\}$, and define $u_0 = u_0(t)$ and $u_0^{-1} = u_0^{-1}(t)$ around x_0 as

$$u_0(t) = c \exp\left(\int_{x_0}^t \omega\right), \quad u_0^{-1}(t) = c^{-1} \exp\left(\int_{x_0}^t -\omega\right);$$

note that the product of them is identically 1 around x_0. For x_i such that $n_i \geq 2$, there are $n_i - 1$ sectors $S^+_{i;1}, \ldots, S^+_{i;n_i-1}$ and $n_i - 1$ sectors $S^-_{i;1}, \ldots, S^-_{i;n_i-1}$ in a small neighborhood U_i of x_i such that

$$\lim_{t \to x_i, t \in S^+_{i;k}} u_0(t) = 0, \qquad \lim_{t \to x_i, t \in S^-_{i;k}} u_0^{-1}(t) = 0,$$

respectively, where u_0 is the analytic continuation of u_0 defined around x_0 along a path from x_0 to a point near x_i. We arrange them as in Figure 1.

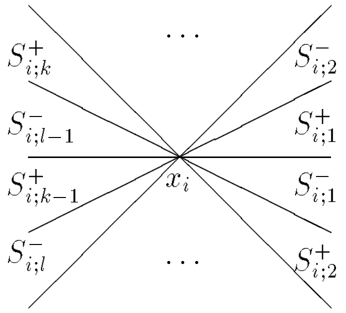

Figure 1: Sectors.

Note that $S^+_{i;1},\ldots,S^+_{i;n_i-1}$ are arranged clockwise and that $S^-_{i;1},\ldots,S^-_{i;n_i-1}$ are arranged counterclockwise (cf. Theorem 4.4).

Let $\rho^+_{i;k}(t)$ be a path from x_i to t in the sector $S^+_{i,k}$. We assign a branch of $u_{\rho^+_{i;k}(t)}(s)$ on $\rho^+_{i;k}(t)$ by the integral

$$\exp\left(\int_t^s \omega\right)$$

along the path from t to $s \in \rho_{i;k}(t)$ in the path $-\rho_{i;k}(t)$. The formal product

$$\gamma^+_{i;k}(t) = \rho^+_{i;k}(t) \otimes u_{\rho^+_{i;k}(t)}(s) \tag{5}$$

is an element of $C^\omega_1(X)$ such that

$$\partial_\omega(\rho^+_{i;k}(t) \otimes u_{\rho^+_{i;k}(t)}) = t \otimes u_{\rho_{i;k}(t)}(t) - x_i \otimes 0 = t \otimes 1.$$

Similarly, we have an element

$$\gamma^-_{i;k}(t) = \rho^-_{i;k}(t) \otimes u^{-1}_{\rho^-_{i;k}(t)}(s) \in C^{-\omega}_1(X) \tag{6}$$

such that

$$\partial_{-\omega}(\rho^-_{i;k}(t) \otimes u^{-1}_{\rho^-_{i;k}(t)}) = t \otimes u^{-1}_{\rho_{i;k}(t)}(t) - x_i \otimes 0 = t \otimes 1.$$

For x_i such that $n_i = 1$, let $\rho^+_{i;1}$ be a loop turning around x_i counterclockwise and terminating at t, and let $\rho^-_{i;1} = -\rho^+_{i;1}$. We assign a branch of $u_{\rho^+_{i;1}(t)}(s)$ on $\rho^+_{i;1}(t)$ and that of $u_{\rho^-_{i;1}(t)}(s)$ on $\rho^-_{i;1}(t)$ by the integrals

$$\exp\left(\int_t^s \omega\right), \quad \exp\left(\int_t^s -\omega\right)$$

along the path from the ending point t to $s \in \rho_k^{\pm}(t)$ in the path $-\rho_{i;1}^{\pm}(t)$, respectively. We can regard

$$\begin{aligned}
\gamma_{i;1}^{+}(t) &= \frac{c_i}{c_i - 1}\rho_{i;1}^{+} \otimes u_{\rho_{i;1}^{+}(t)} \in C_1^{\omega}(X), \quad (7)\\
\gamma_{i;1}^{-}(t) &= \frac{c_i}{c_i - 1}\rho_{i;1}^{-} \otimes u_{\rho_{i;1}^{-}(t)} \in C_1^{-\omega}(X), \quad (8)
\end{aligned}$$

by dividing $\rho_{i;1}^{\pm}$ into two simply connected paths, where $c_i = \exp(2\pi\sqrt{-1}\alpha_{i;1})$. Note that

$$\begin{aligned}
\partial_{\omega}(\gamma_{i;1}^{+}(t)) &= \frac{c_i}{c_i - 1}(t \otimes 1 - t \otimes c_i^{-1}) = t \otimes 1,\\
\partial_{-\omega}(\gamma_{i;1}^{-}(t)) &= \frac{c_i}{c_i - 1}(t \otimes 1 - t \otimes c_i^{-1}) = t \otimes 1.
\end{aligned}$$

Any $\gamma_{i;k}^{\pm}(t)$ can be extended along a path to a general point $t \in X$ so that $\partial_{\pm\omega}(\gamma_{i;k}^{\pm}(t)) = t \otimes 1$. Since

$$\partial_{\pm\omega}(\gamma_{i;k}^{\pm}(t) - \gamma_{j;l}^{\pm}(t)) = 0,$$

$\gamma_{ij;kl}^{\pm} = \gamma_{i;k}^{\pm}(t) - \gamma_{j;l}^{\pm}(t)$ belongs to $H_1(C_{\bullet}^{\pm\omega}(X), \partial_{\pm\omega})$.

Lemma **3.1** *If $\varphi \in H^1(S^{\bullet}(x), \nabla_{\omega})$ satisfies $\langle\varphi, \gamma\rangle = 0$ for all $\gamma \in H_1(C_{\bullet}^{\omega}(X), \partial_{\omega})$, then $\varphi = 0$ in $H^1(S^{\bullet}(x), \nabla_{\omega})$.*

Proof. For i such that $n_i > 1$, we define a function $f_{i;k}$ on $S_{i;k}^{+}$ as

$$f_{i;k}(t) = \langle\varphi, \gamma_{i;k}^{+}(t)\rangle = \int_{\rho_{i;k}^{+}(t)} \exp\left(\int_t^s \omega\right)\varphi(s).$$

Since s is nearer to x_i than to t in $S_{i;k}^{+}$, $\exp\left(\int_t^s \omega\right)$ is bounded as $t \to x_i$ in $S_{i;k}^{+}$, which implies that $f_{i;k}(t)$ is well-defined and rapidly deceases at x_i in $S_{i;k}^{+}$. Since

$$\begin{aligned}
df_{i;k}(t) &= \exp\left(\int_t^t \omega\right)\varphi(t) + \left(\int_{\rho_{i;k}^{+}(t)} \frac{\partial}{\partial t}\left(\exp\left(\int_t^s \omega\right)\right)\varphi(s)\right) dt\\
&= \varphi - f_{i;k}\omega,
\end{aligned}$$

we have $\nabla_{\omega} f_{i;k} = \varphi$. By a suitable choice of a path, we see that as $t \to x_i$ in the neighboring sectors $S_{i;l-1}^{-}$ and $S_{i;l}^{-}$ of $S_{i;k}$, $\exp\left(\int_t^s \omega\right)$ is bounded on $S_{i;k}^{+} \cup S_{i;l-1}^{-} \cup S_{i;l}^{-}$, which implies that $f_{i;k}$ can be extended to $S^-_{i;l-1}$ and $S^-_{i;l}$ and that $f_{i;k}(t)$ rapidly deceases at x_i in $S^+_{i;k} \cup S^-_{i;l-1} \cup S^-_{i;l}$. Indeed, fix a sufficiently small real positive number r and assume that $t \in S^-_{i;l-1}$ and $|t - x_i| \leq r$. Take $t_0 \in S^-_{i;k}$ and $t_1 \in S^+_{i;k}$ so that

$$\arg(t_0 - x_i) = \arg(t - x_i), \quad |t_0 - x_i| = r, \quad |t_1 - x_i| = r.$$

Regard $\rho^+_{i;k}$ as a path connecting the segment $[x_i, t_1]$, the arc from t_1 to t_0 and the segment $[t_0, t]$. When s is not on the segment $[x_i, t_0]$, we can estimate $\exp(\int_t^s \omega)$ by $\exp(\int_t^s \omega) = \exp(\int_t^{t_0} \omega)\exp(\int_{t_0}^s \omega)$ and show the boundedness. When s is on the segment $[x_i, t_0]$, it is easy to estimate $\exp(\int_t^s \omega)$ and show the boundedness.

For i such that $n_i = 1$, we define $f_{i;1}$ on U_i as

$$f_{i;1}(t) = \langle \varphi, \gamma^+_{i;1} \rangle = \frac{c_i}{c_i - 1} \int_{\rho_{i;1}} \exp\left(\int_t^s \omega\right) \varphi(s).$$

By estimating the integral, we can easily show that $f_{i;1}$ rapidly decreases at x_i. Note that $f_{i;1}$ is single-valued on U_i and that $\nabla_\omega f_{i;1} = \varphi$.

Each $f_{i;k}(t)$ can be extended to X. By assumption, all

$$f_{i;k}(t) - f_{j;l}(t) = \langle \varphi, \gamma_{ij;kl} \rangle$$

vanish, which means that the functions $f_{i;k}(t)$ determine $f \in S^0(x)$ satisfying $\nabla_\omega f = \varphi$. q.e.d.

THEOREM **3.2** *The pairing between $H^1(S^\bullet(x), \nabla_\omega)$ and $H_1(C^\omega_\bullet(X), \partial_\omega)$ is perfect.*

Proof. Put $A = \bigcup_{i,j} S^+_{i;j}$ and consider the homology group $H_1(X, A; \mathcal{L}_\omega)$. It is easy to prove by the Mayer-Vietris exact sequence that the dimension of this homology group is equal to $n-2$. Since there exists a surjective linear map from $H_1(X, A; \mathcal{L}_\omega)$ to $H_1(C^\omega_\bullet(X), \partial_\omega)$, we have

$$n' := \dim_{\mathbf{C}} H_1(C^\omega_\bullet(X), \partial_\omega) \leq \dim_{\mathbf{C}} H_1(X, A; \mathcal{L}_\omega) = n - 2.$$

Lemma 3.1 asserts that $\varphi = 0$, provided the map $H_1(C^{\omega}_{\bullet}(X), \partial_\omega) \ni \gamma \mapsto \langle \varphi, \gamma \rangle \in \mathbf{C}$ defined by a vector φ in the $(n-2)$-dimensional vector space $H^1(S^\bullet(x), \nabla_\omega)$ is the zero map. Therefore, $n' \geq n-2$ and hence $n' = n-2$, which implies that the pairing is perfect. q.e.d.

4 Intersection pairings

There is the natural pairing between $S^k(x)$ and $P^{2-k}(x)$ by

$$\int_X \varphi \wedge \psi, \quad \varphi \in S^k(x),\ \psi \in P^{2-k}(x); \tag{9}$$

the integral converges since $\varphi \wedge \psi \in S^2(x)$. Since we have

$$(\nabla_\omega \varphi) \wedge \eta = d(\varphi \wedge \eta) + (-1)^{k+1} \varphi \wedge (\nabla_{-\omega} \eta)$$

for $\varphi \in S^k(x), \eta \in P^{1-k}(x)$, this pairing descends to a pairing $\langle\ ,\ \rangle$ between $H^1(S^\bullet(x), \nabla_\omega)$ and $H^1(P^\bullet(x), \nabla_{-\omega})$, which is called the intersection pairing. By the isomorphisms in Theorem 2.1, this intersection pairing naturally induces a pairing between $H^1(\Omega^\bullet(x), \nabla_{+\omega})$ and $H^1(\Omega^\bullet(x), \nabla_{-\omega})$.

THEOREM **4.1** ([17]) *The intersection number of $\varphi \in H^1(\Omega^\bullet(x), \nabla_\omega)$ and $\psi \in H^1(\Omega^\bullet(x), \nabla_{-\omega})$ is given by*

$$\langle \varphi, \psi \rangle = 2\pi\sqrt{-1} \sum_{i=1}^{m} \mathrm{Res}_{t=x_i}(G_i^1 \cdot \psi),$$

where G_i^1 is a sufficiently large finite part of the formal Laurent series solution G for the equation $\nabla_\omega G = \varphi$ at x_i in (2).

Proof. The explicit form of the image φ under the isomorphism $\iota_\omega : H^1(\Omega^\bullet(x), \nabla_\omega) \to H^1(S^\bullet(x), \nabla_\omega)$ (see (3), (4)) yields

$$\langle \varphi, \psi \rangle = \int_X \left(\varphi - \sum_{i=1}^{m} (h_i \cdot \nabla_\omega(f_i) + f_i \cdot dh_i) \right) \wedge \psi.$$

Note that

$$\left(\varphi - \sum_{i=1}^{m}(h_i \cdot \nabla_\omega(f_i))\right) \wedge \psi \in S^2(x)$$

and that its support is $\bigcup_{i=1}^m U_i$. For any $\varepsilon > 0$, we have

$$\left|\int_{U_i} (\varphi - (h_i \cdot \nabla_\omega(f_i))) \wedge \psi\right| < \varepsilon,$$

if we take a sufficiently small U_i.

Since the support of dh_i is $U_i \setminus V_i$, we have

$$\begin{aligned} -\int_X f_i \cdot dh_i \wedge \psi &= -\int_{U_i\setminus V_i} (G_i^1 + F_i) \cdot dh_i \wedge \psi \\ &= -\int_{U_i\setminus V_i} d(h_i \cdot (G_i^1 \cdot \psi)) - \int_{U_i\setminus V_i} dh_i \wedge (F_i \cdot \psi). \end{aligned}$$

The Stokes theorem and the residue theorem together with the property

$$h_i = \begin{cases} 0 & \text{on} \quad \partial U_i, \\ 1 & \text{on} \quad \partial V_i, \end{cases}$$

then imply

$$\begin{aligned} -\int_{U_i\setminus V_i} d(h_i \cdot (G_i^1 \cdot \psi)) &= -\int_{\partial U_i} h_i \cdot (G_i^1 \cdot \psi) + \int_{\partial V_i} h_i \cdot (G_i^1 \cdot \psi) \\ &= \int_{\partial V_i} G_i^1 \cdot \psi = 2\pi\sqrt{-1}\mathrm{Res}_{t=x_i}(G_i^1 \cdot \psi). \end{aligned}$$

On the other hand, since

$$d(h_i \cdot (F_i \cdot \psi)) = dh_i \wedge (F_i \cdot \psi) + h_i d(F_i \cdot \psi),$$

we have

$$\begin{aligned} -\int_{U_i\setminus V_i} dh_i \wedge (F_i \cdot \psi) &= -\int_{U_i\setminus V_i} d(h_i \cdot (F_i \cdot \psi)) + \int_{U_i\setminus V_i} h_i d(F_i \cdot \psi) \\ &= -\int_{\partial U_i \setminus \partial V_i} h_i \cdot (F_i \cdot \psi) + \int_{U_i\setminus V_i} h_i d(F_i \cdot \psi) \\ &= \int_{\partial V_i} F_i \cdot \psi + \int_{U_i\setminus V_i} h_i d(F_i \cdot \psi). \end{aligned}$$

Since $F_i \cdot \psi$ is smooth on U_i for a sufficiently large N, for any $\varepsilon > 0$ we can take a sufficiently small U_i such that

$$\left|\int_{\partial V_i} F_i \cdot \psi\right| < \varepsilon, \quad \left|\int_{U_i\setminus V_i} h_i d(F_i \cdot \psi)\right| < \varepsilon. \qquad \text{q.e.d}$$

The intersection numbers for suitable bases of $H^1(\Omega^\bullet(x), \nabla_{\pm\omega})$ are evaluated in [17]. It is shown in [7] that the determinant of the intersection matrix of $H^1(\Omega^\bullet(x), \nabla_{\pm\omega})$ is

$$(2\pi\sqrt{-1})^{n-2} \prod_{i=\sigma+1}^{m} \frac{1}{\alpha_{i;1}},$$

which implies that the intersection form between the twisted cohomology groups is perfect.

In order to define the intersection pairing between $H_1(C^\omega_\bullet(X), \partial_\omega)$ and $H_1(C^{-\omega}_\bullet(X), \partial_{-\omega})$ by the duality in Theorem 3.2, we introduce spaces of temperate currents as follows. By a semi-norm similar to that for Schwartz's space of rapidly decreasing functions, the space $S^k(x)$ becomes a Fréchet space. The space $\check{S}^{2-k}(x)$ of continuous $\mathbf{C}$-linear functionals on $S^k(x)$ is called the space of temperate currents of degree $2-k$. Taking the dual complex of $(S^\bullet(x), \nabla_\omega)$, we have a complex with differential $\nabla_{-\omega}$:

$$(\check{S}^\bullet(x), \nabla_{-\omega}) : \check{S}^0(x) \xrightarrow{\nabla_{-\omega}} \check{S}^1(x) \xrightarrow{\nabla_{-\omega}} \check{S}^2(x) \longrightarrow 0.$$

Since

$$\int_X \nabla_\omega(\xi) \wedge \eta = (-1)^{k+1} \int_X \xi \wedge \nabla_{-\omega}(\eta), \quad \langle \nabla_\omega(\xi), \gamma \rangle = \langle \xi, \partial_\omega(\gamma) \rangle$$

for $\xi \in S^k(x)$, $\eta \in P^{1-k}(x)$ and $\gamma \in C^\omega_{k+1}(X)$, we have natural inclusions of complexes

$$(P^\bullet(x), \nabla_{-\omega}) \subset (\check{S}^\bullet(x), \nabla_{-\omega}), \quad (C^\omega_\bullet(X), \partial_\omega) \subset (\check{S}^\bullet(x), \nabla_{-\omega}),$$

which induce the maps

$$\iota_1 \quad : \quad H^1(P^\bullet(x), \nabla_{-\omega}) \to H^1(\check{S}^\bullet(x), \nabla_{-\omega}), \tag{10}$$

$$\iota_2 \quad : \quad H_1(C^\omega_\bullet(X), \partial_\omega) \to H^1(\check{S}^\bullet(x), \nabla_{-\omega}), \tag{11}$$

respectively, where $H^1(\check{S}^\bullet(x), \nabla_{-\omega})$ is the first cohomology group of the complex $(\check{S}^\bullet(x), \nabla_{-\omega})$.

Theorem 4.2 *The maps ι_1 and ι_2 are isomorphisms.*

Proof. It was proved in [14, p.81, i)] that

$$H^1(\check{S}^\bullet(x), \nabla_{-\omega}) \simeq H^1(\Omega^\bullet(x), \nabla_{-\omega}).$$

Therefore, the map ι_1 is an isomorphism by virtue of Theorem 2.1.

The injectivity of ι_2 follows from the perfectness (Theorem 3.2) of the pairing between the homology and cohomology groups. Since the dimensions of both sides agree, the map ι_2 is an isomorphism. q.e.d.

By Theorems 2.1 and 4.2, we get the isomorphisms

$$\jmath^+ \quad : \quad H_1(C_\bullet^{-\omega}(X), \partial_{-\omega}) \to H^1(S^\bullet(x), \nabla_\omega), \tag{12}$$

$$\jmath^- \quad : \quad H_1(C_\bullet^{\omega}(X), \partial_{\omega}) \to H^1(P^\bullet(x), \nabla_{-\omega}). \tag{13}$$

The intersection number of $\gamma^+ \in H_1(C_\bullet^\omega(X), \partial_\omega)$ and $\gamma^- \in H_1(C_\bullet^{-\omega}(X), \partial_{-\omega})$ is defined by

$$\langle \gamma^+, \gamma^- \rangle = \langle \jmath^-(\gamma^+), \jmath^+(\gamma^-) \rangle = \int_X \jmath^-(\gamma^+) \wedge \jmath^+(\gamma^-).$$

Theorem 4.3 *Let*

$$\gamma^+ = \sum_\nu b_\nu \rho_\nu^+ \otimes u_{\rho_\nu^+}(t) \in H_1(C_\bullet^\omega(X), \partial_\omega),$$
$$\gamma^- = \sum_\mu b_\mu \rho_\mu^- \otimes u^{-1}_{\rho_\mu^-}(t) \in H_1(C_\bullet^{-\omega}(X), \partial_{-\omega}).$$

If the set $\bigcup_{\nu,\mu}(\rho_\nu^+ \cap \rho_\mu^-)$ is finite and ρ_ν^+ and ρ_μ^- intersect transversally at each point of $\rho_\nu^+ \cap \rho_\mu^- \cap X$, then the intersection number $\langle \gamma^+, \gamma^- \rangle$ is equal to

$$\langle \gamma^+, \gamma^- \rangle = \sum_{\mu,\nu} \sum_{v \in \rho_\nu^+ \cap \rho_\mu^- \cap X} b_\nu b_\mu [u_{\rho_\nu^+}(t)]_{t=v} [u^{-1}_{\rho_\mu^-}(t)]_{t=v} I_v(\rho_\nu^+, \rho_\mu^-),$$

where $I_v(\rho_\nu^+, \rho_\mu^-)$ is the topological intersection number of ρ_ν^+ and ρ_μ^- at $v \in X$.

Proof. Let δ_Δ be a delta r-current which has support on Δ. Then, we have

$$F_{\gamma^+} = \sum_\nu b_\nu \delta_{\rho_\nu^+} u_{\rho_\nu^+}, \quad F_{\gamma^-} = \sum_\mu b_\mu \delta_{\rho_\mu^-} u_{\rho_\mu^-}.$$

Let reg be the regularization

$$\mathrm{reg} : H^1(\check{S}^\bullet(x), \nabla_{-\omega}) \xrightarrow{\sim} H^1(\mathcal{S}^\bullet(x), \nabla_{-\omega}).$$

Then the intersection number $\langle \gamma^+, \gamma^- \rangle$ is equal to

$$\int_X \mathrm{reg}(F_{\gamma^+}) \wedge \mathrm{reg}(F_{\gamma^-}).$$

Hence we are going to evaluate this integral.

If we regard the operator $\nabla_{\pm\omega}$ as an operator on the $2r$-dimensional real manifold X ($r = 1$), it is holonomic at degree $r-1$ and hypo-elliptic (resp. hypo-analytic [8, Theorem 4.3.3]) on X. Indeed, for any current F of degree r and G of degree $r-1$, if $\nabla_{\pm\omega} G = F$ and F is smooth (resp. real analytic) at a point p, then G is also smooth (resp. real analytic) at p. Moreover, the singularity spectrum of G is contained in that of F. Hence, when $\mathrm{reg}(F_{\gamma^+}) = F_{\gamma^+} + \nabla_{-\omega} G_{\gamma^+}$ and $\mathrm{reg}(F_{\gamma^-}) = F_{\gamma^-} + \nabla_\omega G_{\gamma^-}$, the wedge product of G_{γ^+} and G_{γ^-} is well-defined. We note that $\varphi[1] = \langle \varphi, 1 \rangle$ does not always exist for a temperate 2-current, because 1 is not a rapidly decreasing 0-form. Therefore, to evaluate $\int_X \mathrm{reg}(F_{\gamma^+}) \wedge \mathrm{reg}(F_{\gamma^-})$ through evaluation of integrals of currents, we need a more precise description of $G_{\gamma^\pm}$.

By using the Heaviside function, we can express a solution G_{γ^+} of $\mathrm{reg}(F_{\gamma^+}) = F_{\gamma^+} + \nabla_{-\omega} G_{\gamma^+}$ as

$$G_{\gamma^+} = u_{\gamma^+} v_{\gamma^+}, \quad u_{\gamma^+} \in S^0(x),\ v_{\gamma^+} \in \check{S}^0(x). \tag{14}$$

Consider now the wedge product of $\mathrm{reg}(F_{\gamma^+}) = F_{\gamma^+} + \nabla_{-\omega} G_{\gamma^+}$ and $\mathrm{reg}(F_{\gamma^-}) = F_{\gamma^-} + \nabla_\omega G_{\gamma^-}$. Then we have

$$\mathrm{reg}(F_{\gamma^+}) \wedge \mathrm{reg}(F_{\gamma^-}) = F_{\gamma^+} \wedge F_{\gamma^-} + F_{\gamma^+} \wedge \nabla_\omega G_{\gamma^-} + \nabla_{-\omega} G_{\gamma^+} \wedge F_{\gamma^-} + \nabla_{-\omega} G_{\gamma^+} \wedge \nabla_\omega G_{\gamma^-}.$$

It follows from (14) that all terms on the right hand side can be expressed as

(a rapidly decreasing smooth 0-form) $\wedge$ (a temperate 2-current).

Hence, the integrals of the 2-currents exist. Therefore, by the Stokes theorem (cf. Kita-Yoshida [11, 1.5]), we have

$$\begin{aligned}\int_X \mathrm{reg}(F_{\gamma^+}) \wedge \mathrm{reg}(F_{\gamma^-}) &= \int_X F_{\gamma^+} \wedge F_{\gamma^-} + \int_X F_{\gamma^+} \wedge \nabla_\omega G_{\gamma^-} \\ &\quad + \int_X \nabla_{-\omega} G_{\gamma^+} \wedge F_{\gamma^-} + \int_X \nabla_{-\omega} G_{\gamma^+} \wedge \nabla_\omega G_{\gamma^-} \\ &= \int_X F_{\gamma^+} \wedge F_{\gamma^-},\end{aligned}$$

which is equal to

$$\sum_{\mu,\nu} \sum_{v \in \rho_\nu^+ \cap \rho_\mu^- \cap X} b_\nu b_\mu [u_{\rho_\nu^+}(t)]_{t=v} [u^{-1}_{\rho_\mu^-}(t)]_{t=v} I_v(\rho_\nu^+, \rho_\mu^-). \qquad \text{q.e.d}$$

We introduce the following explicit cycles for twisted homology groups. Assume that the base point x_0 is in the upper half space $\mathbf{H}$,

$$x_i \in \mathbf{R}, \quad x_1 < x_2 < \cdots < x_m$$

and that the interior of the closure of $S^+_{i;1} \cup S^-_{i;1}$ in X contains the set

$$L_i = \{t \in U_i \mid \mathrm{Re}(t - x_i) > 0,\ \mathrm{Im}(t - x_i) = 0\}$$

for i such that $n_i > 1$. We define $\gamma^\pm_{i;k}(x_0)$ by the continuation of $\gamma^\pm_{i;k}(t)$ in (5), (6), (7) along a path from $t \in S^\pm_{i;k}$ passing through $U_i \setminus L_i$ to a point in $\mathbf{H} \cap S^+_{i;1}$ and going to x_0 in $\mathbf{H}$. For i such that $n_i > 1$, we define $\tilde{\gamma}^+_{i;1}(x_0)$ by the continuation of $\gamma^+_{i;1}(t)$ along a path from $t \in S^+_{1;k}$ turning around x_i counterclockwise in U_i and going to x_0 in $\mathbf{H} \cup S^+_{i;1}$, and $\tilde{\gamma}^-_{i;1}(x_0)$ by the continuation of $\gamma^-_{i;1}(t)$ along a path from $t \in S^-_{1;k}$ traversing L_i to a point in $\mathbf{H} \cap S^+_{i;1}$ and going to x_0 in $\mathbf{H}$. The topological path $\rho^\pm_{i;k}(x_0)$ of $\gamma^\pm_{i;k}(x_0)$ is as given in Figure 2. We define

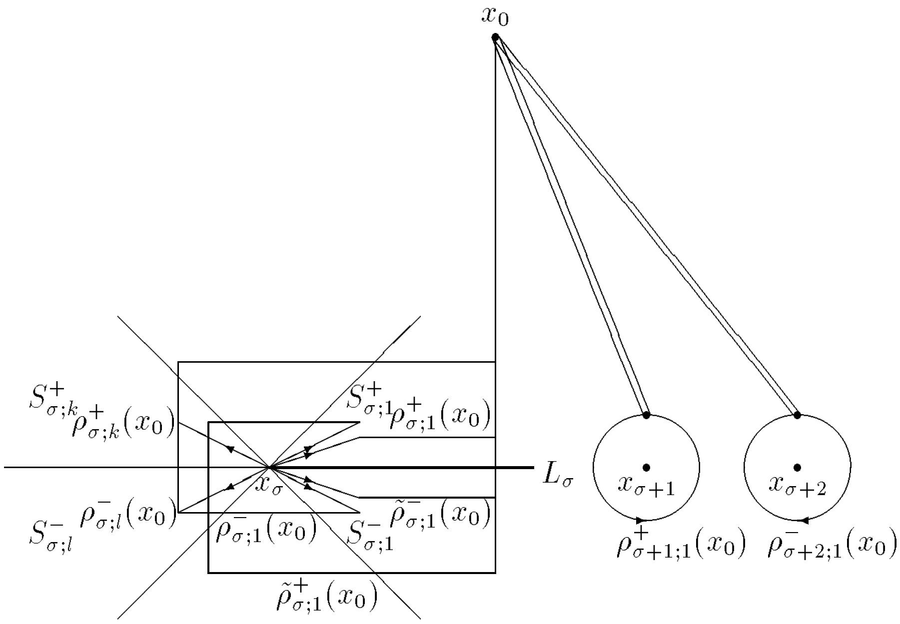

Figure 2: Paths.

$$\overbrace{\gamma^\pm_{1;1},\ \cdots,\ \gamma^\pm_{1;n_1-1}}^{n_1-1},\quad \overbrace{\gamma^\pm_{2;1},\ \cdots,\ \gamma^\pm_{2;n_2-1}}^{n_2-1},\quad \cdots,\quad \overbrace{\gamma^\pm_{m;1},\ \cdots,\ \gamma^\pm_{m;n_m-1}}^{n_m-1},$$

$$\overbrace{\gamma^\pm_{1;n_1},\ \gamma^\pm_{2;n_2},\ \cdots,\ \gamma^\pm_{m-1;n_{m-1}}}^{m-1}$$

as

$$\gamma^\pm_{i;k} = \begin{cases} \gamma^\pm_{i;1}(x_0) - \tilde{\gamma}^\pm_{i;1}(x_0), & 1 \le i \le \sigma,\ k = 1, \\ \gamma^\pm_{i;1}(x_0) - \gamma^\pm_{i;k}(x_0), & 1 \le i \le \sigma,\ 2 \le k \le n_i - 1, \\ \gamma^\pm_{i+1;1}(x_0) - \gamma^\pm_{i;1}(x_0), & 1 \le i \le m-1,\ k = n_i. \end{cases}$$

THEOREM **4.4** *The intersection matrix* $I_{\mathrm{h}} = \langle \gamma^+_{i;k}, \gamma^-_{j;l} \rangle$ *is*

$$\begin{pmatrix} \Xi_1 & & & & D_1 \\ & \Xi_2 & & & D_2 \\ & & \ddots & & \vdots \\ & & & \Xi_\sigma & D_\sigma \\ {}^tD_1 & {}^tD_2 & \cdots & {}^tD_\sigma & G \end{pmatrix},$$

whose entries are given as follows: the (n_i-1, n_i-1)*-matrix* Ξ_i *is*

$$\begin{pmatrix} c_i-1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

the $(n_i-1, m-1)$*-matrix* D_i *is*

$$\begin{array}{c} \begin{array}{cccccc} & & {\scriptstyle (i-1)} & {\scriptstyle i} & & \end{array} \\ \begin{pmatrix} 0 & \cdots & -1 & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array},$$

and the $(m-1, m-1)$*-matrix* G *is*

$$G = \begin{array}{c} \\ \\ \\ \\ {\scriptstyle \sigma} \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccc} & & & & {\scriptstyle \sigma} & & & \end{array} \\ \begin{pmatrix} -1 & 1 & & & \vdots & & & \\ & -1 & 1 & & \vdots & & & \\ & & \ddots & \ddots & \vdots & & & \\ & & & -1 & 1 & & & \\ \cdots & \cdots & \cdots & \cdots & \frac{c_{\sigma+1}}{1-c_{\sigma+1}} & \frac{-c_{\sigma+1}}{1-c_{\sigma+1}} & & \\ & & & & \frac{-1}{1-c_{\sigma+1}} & \frac{1-c_{\sigma+1}c_{\sigma+2}}{(1-c_{\sigma+1})(1-c_{\sigma+2})} & \frac{-c_{\sigma+2}}{1-c_{\sigma+2}} & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix} \end{array},$$

where $c_i = \exp(2\pi\sqrt{-1}\alpha_{i;1})$ *and* $\sigma = \#\{j \mid n_j > 1\}$.

Proof. By applying Theorem 4.3, we have the desired intersection matrix. q.e.d.

It is shown in [7] that the $(1,1)$-minor of I_{h} is

$$\prod_{i=\sigma+1}^{m} \frac{1}{d_i},$$

which implies that $\{\gamma^+_{i;k}\}_{i;k}$ and $\{\gamma^-_{i;k}\}_{i;k}$ form bases of $H_1(C^{\omega}_{\bullet}(X), \partial_{\omega})$ and $H_1(C^{-\omega}_{\bullet}(X), \partial_{-\omega})$, respectively.

5 Twisted period relations

Let $\{\varphi^{\pm}_{\mu}\}$ and $\{\gamma^{\pm}_{\mu}\}$ be bases of $H^1(\Omega^{\bullet}(x), \nabla_{\pm\omega})$ and $H_1(C^{\pm\omega}_{\bullet}(X), \partial_{\pm\omega})$, respectively. We define four $(n-2, n-2)$-matrices as

$$\Pi^+ = \langle \varphi^+_{\mu}, \gamma^+_{\nu} \rangle_{\mu,\nu}, \quad \Pi^- = \langle \varphi^-_{\mu}, \gamma^-_{\nu} \rangle_{\mu,\nu}, \quad I_{\mathrm{ch}} = \langle \varphi^+_{\mu}, \varphi^-_{\nu} \rangle_{\mu,\nu}, \quad I_{\mathrm{h}} = \langle \gamma^+_{\mu}, \gamma^-_{\nu} \rangle_{\mu,\nu}.$$

The naturality of the pairings between the twisted cohomology group and the twisted homology group implies the following (cf. [1, Theorem 2]).

THEOREM **5.1** *We have twisted period relations with respect to* $\pm\omega$:

$$\Pi^+ \, {}^t I_{\mathrm{h}}^{-1} \, {}^t\Pi^- = I_{\mathrm{ch}}, \quad i.e., \quad {}^t\Pi^- \, I_{\mathrm{ch}}^{-1} \, \Pi^+ = {}^t I_{\mathrm{h}}.$$

Since I_{ch} and I_{h} can be computed explicitly, the above identities yield quadratic relations among confluent hypergeometric functions.

5.1 The gamma function $\Gamma(\alpha)$

The gamma function is defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha} \frac{dt}{t}$$

for $\mathrm{Re}(\alpha) > 0$. Let us derive the inversion formula for the gamma function as a twisted period relation by using Theorems 4.1 and 4.3.

We put

$$u(t) = e^{-t} t^{\alpha}, \quad \omega = -dt + \alpha \frac{dt}{t} \ (\alpha \neq 0), \quad (n_1, n_2) = (2, 1), \quad (x_1, x_2) = (\infty, 0).$$

Since $n_1 + n_2 = 3$, $H^1(\Omega^{\bullet}(X), \nabla_{\omega})$ and $H_1(C^{\omega}_{\bullet}(X), \partial_{\omega})$ are 1-dimensional. We define a branch $u_0(t)$ of $u(t)$ around $t = 1$ as

$$u_0(t) = \frac{1}{e} \exp\left(\int_1^t \omega \right);$$

note that

$$u_0^{-1}(t) = e\exp\left(\int_1^t -\omega\right)$$

around $t = 1$. By Cauchy's integral formula, we have

$$\Gamma(\alpha) = \frac{1}{1 - e^{-2\pi\sqrt{-1}\alpha}} \int_C e^{-t} t^\alpha \frac{dt}{t}$$

for $\alpha \notin \mathbf{Z}$, where C is described in Figure 3 and the argument of t in C belongs to $[-2\pi, 0]$. We regard the integral as the pairing of

$$\varphi^+ = dt/t \in H^1(\Omega^\bullet(x), \nabla_{+\omega})$$

and

$$\gamma^+ = \frac{1}{1 - e^{-2\pi\sqrt{-1}\alpha}} C \otimes u_0(t) \in H_1(C_\bullet^\omega(X), \partial_\omega).$$

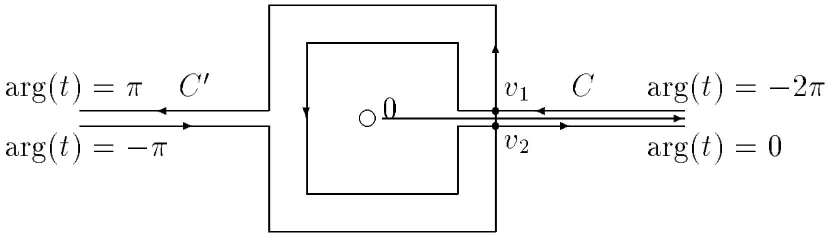

Figure 3: Cycles.

Put $\varphi^- = dt/t \in H^1(\Omega^\bullet(x), \nabla_{-\omega})$ and define a twisted cycle g^- by $\gamma^- = C' \otimes u_0^{-1}(t) \in H_1(C_\bullet^{-\omega}(X), \partial_{-\omega})$, where C' is as described in Figure 3 and the argument of t belongs to $[-\pi, \pi]$. Apply the change of coordinate $t = e^{\pi\sqrt{-1}} s$ to the integral $\langle \varphi^-, \gamma^- \rangle = \int_{C'} e^t t^{-\alpha} dt/t$. Then we have

$$\begin{aligned}
\langle \varphi^-, \gamma^- \rangle &= \int_C e^{-s} e^{-\alpha\pi\sqrt{-1}} s^{-\alpha} \frac{ds}{s} \\
&= e^{-\alpha\pi\sqrt{-1}}(1 - e^{-2\pi\sqrt{-1}(-\alpha)})\Gamma(-\alpha) \\
&= -2\sqrt{-1}\sin(\pi\alpha)\Gamma(-\alpha).
\end{aligned}$$

We evaluate $\langle \varphi^+, \varphi^- \rangle$ by Theorem 4.1. We need to solve

$$\nabla_\omega G = dG - Gdt + \alpha G\frac{dt}{t} = \varphi^+ = \frac{dt}{t}$$

around $t = x_1 = \infty$ and $t = x_2 = 0$. The formal solutions G_1 and G_2 around $t = \infty$ and $t = 0$ are expressed as

$$\begin{aligned} G_1 &= -s - (\alpha - 1)s^2 - (\alpha - 1)(\alpha - 2)s^3 - \cdots, \\ G_2 &= \frac{1}{\alpha} + \frac{1}{\alpha(\alpha+1)}t + \frac{1}{\alpha(\alpha+1)(\alpha+2)}t^2 + \cdots, \end{aligned}$$

respectively, where $s = 1/t$ is a local coordinate around ∞. Theorem 4.1 implies that

$$\langle \varphi^+, \varphi^- \rangle = 2\pi\sqrt{-1}(\mathrm{Res}_{s=0} G_1\frac{-ds}{s} + \mathrm{Res}_{t=0} G_2\frac{dt}{t}) = 2\pi\sqrt{-1}(0 + \frac{1}{\alpha}) = \frac{2\pi\sqrt{-1}}{\alpha}.$$

Next, we evaluate $\langle \gamma^+, \gamma^- \rangle$ from Theorem 4.3. We have $C \cap C' = \{v_1, v_2\}$. (see Figure 3). The topological intersection number of C and C' at v_1 is -1 and that at v_2 is 1. Note that

$$u_0(t)u_0^{-1}(t)|_{t=v_1} = e^{-2\pi\sqrt{-1}\alpha}, \quad u_0(t)u_0^{-1}(t)|_{t=v_2} = 1.$$

Then the intersection number $\langle \gamma^+, \gamma^- \rangle$ is

$$\frac{1}{1 - e^{-2\pi\sqrt{-1}\alpha}}(-e^{-2\pi\sqrt{-1}\alpha} + 1) = 1.$$

The twisted period relation for $\varphi^\pm$ and $\gamma^\pm$ is

$$\begin{aligned} \langle \varphi^+, \gamma^+ \rangle\, 1\, \langle \varphi^-, \gamma^- \rangle &= 2\pi\sqrt{-1}\frac{1}{\alpha}, \\ \Gamma(\alpha)\{-2\sqrt{-1}\sin(\pi\alpha)\Gamma(-\alpha)\} &= 2\pi\sqrt{-1}\frac{1}{\alpha}, \end{aligned}$$

which is nothing but the inversion formula for the gamma function:

$$\Gamma(\alpha)\Gamma(1 - \alpha) = \frac{\pi}{\sin(\pi\alpha)}.$$

5.2 The integral $\int_{-\infty}^{+\infty} e^{-t^2/2}dt$

Let $\omega = -tdt$ and $n_1 = n = 3$, $x_1 = \infty$. The spaces $H^1(\Omega^\bullet(x), \nabla_{\pm\omega})$ and $H_1(C_\bullet^{\pm\omega}(X), \partial_{\pm\omega})$ are 1-dimensional. Let $\varphi^\pm = dt$ and

$$\gamma^+ = [-\infty, \infty] \otimes e^{-t^2/2}, \quad \gamma^- = [-\sqrt{-1}\infty, \sqrt{-1}\infty] \otimes e^{t^2/2}.$$

The intersection number $\langle dt, dt\rangle$ is $2\pi\sqrt{-1}$ as given in Theorem 4.1 and [17], and Theorem 4.3 implies $\langle \gamma^+, \gamma^-\rangle = 1$. Since

$$\langle dt, \gamma^+\rangle = \int_{-\infty}^{+\infty} e^{-t^2/2}dt, \ \langle dt, \gamma^-\rangle = \int_{-\sqrt{-1}\infty}^{+\sqrt{-1}\infty} e^{t^2/2}dt = \sqrt{-1}\int_{-\infty}^{+\infty} e^{-t^2/2}dt,$$

we have the twisted period relation

$$\left(\int_{-\infty}^{\infty} e^{-t^2/2}dt\right) \cdot 1 \cdot \left(\sqrt{-1}\int_{-\infty}^{\infty} e^{-t^2/2}dt\right) = 2\pi\sqrt{-1},$$

which yields the identity $\int_{-\infty}^{+\infty} e^{-t^2/2}dt = \sqrt{2\pi}$.

5.3 The Bessel function $(n = 4)$

The Bessel function is defined by the power series

$$J_a(z) = \left(\frac{z}{2}\right)^a \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(a+k+1)} \left(\frac{z}{2}\right)^{2k},$$

where $z \in \{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$, the argument of $z/2$ is in $(-\pi/2, \pi/2)$, and $a \in \mathbf{C}$. It is known that $J_a(z)$ satisfies the Bessel differential equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{a^2}{z}\right) = 0,$$

and that $J_a(z)$ admits the integral representation

$$J_a(z) = \frac{1}{2\pi\sqrt{-1}} \int_{C'} \exp\left(\frac{z}{2}\left(t - \frac{1}{t}\right)\right) t^{-a}\frac{dt}{t},$$

where C' is as in Figure 3 and the argument of t on C' is in $[-\pi, \pi]$. By putting

$$u(t) = \exp\left(\frac{z}{2}\left(t - \frac{1}{t}\right)\right) t^a, \quad \omega = \left(\frac{z}{2} + \frac{z}{2}\frac{1}{t^2} - a\frac{1}{t}\right) dt,$$

$$(n_1, n_2) = (2,2), \quad (x_1, x_2) = (\infty, 0),$$

we regard $J_a(z)$ as the pairing $\langle \varphi_1^+, \gamma_1^+ \rangle$, where

$$\begin{aligned} \varphi_1^+ &= \frac{1}{2\pi\sqrt{-1}} \frac{dt}{t} \in H^1(\Omega^\bullet(x), \nabla_\omega), \\ \gamma_1^+ &= C' \otimes u(t) \in H_1(C_\bullet^\omega(X), \partial_\omega). \end{aligned}$$

Take

$$\begin{aligned} \varphi_2^+ &= \frac{dt}{2\pi\sqrt{-1}} \in H^1(\Omega^\bullet(x), \nabla_\omega), \\ \varphi_1^- &= \frac{1}{2\pi\sqrt{-1}} \frac{dt}{t} \in H^1(\Omega^\bullet(x), \nabla_{-\omega}), \\ \varphi_2^- &= \frac{1}{2\pi\sqrt{-1}} \frac{dt}{t^2} \in H^1(\Omega^\bullet(x), \nabla_{-\omega}), \\ \gamma_1^- &= C \otimes u^{-1}(t) \in H_1(C_\bullet^{-\omega}(X), \partial_{-\omega}), \end{aligned}$$

where C is as in Figure 3 and the argument of t on C is in $[-2\pi, 0]$. By results in Theorem 4.1 and [17], the intersection matrix $\langle \varphi_i^+, \varphi_j^- \rangle_{ij}$ is

$$\frac{1}{2\pi\sqrt{-1}} \begin{pmatrix} 0 & 2/z \\ -2/z & 0 \end{pmatrix}.$$

We have

$$I_{v_1}(C', C) = 1, \quad I_{v_2}(C', C) = -1,$$

$$u(v_1)u^{-1}(v_1) = \exp(-2\pi\sqrt{-1}a), \quad u(v_2)u^{-1}(v_2) = 1.$$

Thus the intersection number of γ_1^+ and γ_1^- is $\exp(-2\pi\sqrt{-1}a) - 1$. The twisted period relation

$${}^t\Pi^- \ I_{\mathrm{ch}}^{-1} \ \Pi^+ = {}^tI_{\mathrm{h}}$$

implies that

$$\begin{aligned} & 2\pi\sqrt{-1} \left(\frac{z}{2}\right) \left(\langle \varphi_1^-, \gamma_1^- \rangle, \langle \varphi_2^-, \gamma_1^- \rangle \right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle \varphi_1^+, \gamma_1^+ \rangle \\ \langle \varphi_2^+, \gamma_1^+ \rangle \end{pmatrix} \\ = \ & \exp(-2\pi\sqrt{-1}a) - 1. \end{aligned}$$

Note that $\langle\varphi_2^+, \gamma_1^+\rangle = J_{a-1}(z)$. Since

$$\int_C \exp\left(-\frac{z}{2}\left(t-\frac{1}{t}\right)\right) t^a \frac{dt}{t} = \exp(-\pi\sqrt{-1}a)\int_{C'} \exp\left(\frac{z}{2}\left(s-\frac{1}{s}\right)\right) s^a \frac{ds}{s},$$

$$\int_C \exp\left(-\frac{z}{2}\left(t-\frac{1}{t}\right)\right) t^a \frac{dt}{t^2} = \exp(-\pi\sqrt{-1}a)\int_{C'} \exp\left(\frac{z}{2}\left(s-\frac{1}{s}\right)\right) s^a e^{\pi\sqrt{-1}} \frac{ds}{s^2},$$

by the change of variable $t = \exp(-\pi\sqrt{-1})s$, we have

$$\begin{aligned} \langle\varphi_1^-, \gamma_1^-\rangle &= \exp(-\pi\sqrt{-1}a)J_{-a}(z), \\ \langle\varphi_2^-, \gamma_1^-\rangle &= -\exp(-\pi\sqrt{-1}a)J_{-a+1}(z). \end{aligned}$$

Hence we have a quadratic identity among the Bessel functions

$$J_a(z)J_{-a+1}(z) + J_{a-1}(z)J_{-a}(z) = \frac{2\sin(\pi a)}{\pi z},$$

which is called Lommel's formula. For $a = 1/2$, this formula is equivalent to $\sin^2 x + \cos^2 x = 1$.

5.4 A confluent hypergeometric function of two variables ($n = 5$)

The function $\Phi_2(b_1, b_2, c; z_1, z_2)$ defined by the power series

$$\sum_{k_1,k_2=0}^{\infty} \frac{(b_1; k_1)(b_2; k_2)}{(c; k_1+k_2)k_1!k_2!} z_1^{k_1} z_2^{k_2},$$

which converges in $\mathbf{C}^2$, is one of the confluent hypergeometric functions derived from Appell's hypergeometric function $F_1(a, b_1, b_2, c; z_1, z_2)$. Here $(b; k)$ stands for $b(b+1)(b+2)\cdots(b+k-1)$. This function admits the integral representation

$$\frac{1}{\Gamma(1-c)(1-\exp(2\pi\sqrt{-1}c))}\int_C t^{b_1+b_2-c}(t-z_1)^{-b_1}(t-z_2)^{-b_2}e^{-t}dt,$$

where C is a path from $+\infty$ turning along a large circle containing z_1, z_2 and 0 counterclockwise and going to $+\infty$, and the arguments of $t, (t-z_1)$ and $(t-z_2)$ is near 0 around the end point of C (see Figure 3). Put

$$\begin{aligned} \omega &= d\log\left(t^{b_1+b_2-c}(t-z_1)^{-b_1}(t-z_2)^{-b_2}e^{-t}\right) \\ &= \left(\frac{b_1+b_2-c}{t} - \frac{b_1}{t-z_1} - \frac{b_2}{t-z_2} - 1\right)dt, \end{aligned}$$

$$(n_1,n_2,n_3,n_4)=(2,1,1,1),\quad (x_1,x_2,x_3,x_4)=(\infty,0,z_1,z_2),$$

$$\varphi_1^+=dt,\quad \varphi_2^+=\frac{dt}{t-z_1},\quad \varphi_3^+=\frac{dt}{t-z_2},$$

$$\varphi_1^-=\frac{dt}{t},\quad \varphi_2^-=\frac{dt}{t}-\frac{dt}{t-z_1}=\frac{-z_1dt}{t(t-z_1)},\quad \varphi_3^-=\frac{dt}{t}-\frac{dt}{t-z_2}=\frac{-z_2dt}{t(t-z_2)}.$$

We have

$$I_{\mathrm{ch}}=2\pi\sqrt{-1}\begin{pmatrix}1&0&0\\0&-\frac{1}{b_1}&0\\0&0&-\frac{1}{b_2}\end{pmatrix},\quad I_{\mathrm{ch}}^{-1}=\frac{1}{2\pi\sqrt{-1}}\begin{pmatrix}1&0&0\\0&-b_1&0\\0&0&-b_2\end{pmatrix}.$$

Let γ_1^+ be the element of $H_1(C_\bullet^\omega(X),\partial_\omega)$ defined by

$$\frac{1}{\Gamma(1-c)(1-\exp(2\pi\sqrt{-1}c))}C\otimes t^{b_1+b_2-c}(t-z_1)^{-b_1}(t-z_2)^{-b_2}e^{-t},$$

where the arguments of $t,(t-z_1)$ and $(t-z_2)$ is near 0 around the end point of C. Let γ_1^- be the element of $H_1(C_\bullet^{-\omega}(X),\partial_{-\omega})$ defined by

$$\frac{1}{\Gamma(c)(1-\exp(-2\pi\sqrt{-1}c))}C'\otimes t^{-b_1-b_2+c}(t-z_1)^{b_1}(t-z_2)^{b_2}e^{t},$$

where C' is a path from $-\infty$ turning along a large circle containing z_1,z_2 and 0 counterclockwise and going to $-\infty$, and the arguments of $t,(t-z_1)$ and $(t-z_2)$ is near π around the end point of C' (see Figure 3).

It is easy to see that

$$\begin{aligned}\langle\varphi_1^+,\gamma_1^+\rangle &= \Phi_2(b_1,b_2,c;z_1,z_2),\\ \langle\varphi_2^+,\gamma_1^+\rangle &= -\frac{1}{c}\Phi_2(b_1+1,b_2,c+1;z_1,z_2),\\ \langle\varphi_3^+,\gamma_1^+\rangle &= -\frac{1}{c}\Phi_2(b_1,b_2+1,c+1;z_1,z_2).\end{aligned}$$

By putting $t=\exp(\pi\sqrt{-1})s$, we have

$$\begin{aligned}&\langle\varphi_1^-,\gamma_1^-\rangle\\ =&\ \frac{1}{\Gamma(c)(1-\exp(-2\pi\sqrt{-1}c))}\int_{C'}t^{-b_1-b_2+c}(t-z_1)^{b_1}(t-z_2)^{b_2}e^t\frac{dt}{t},\\ =&\ \frac{1}{\Gamma(c)(1-\exp(-2\pi\sqrt{-1}c))}\int_{C}(-s)^{-b_1-b_2+c}(-s-z_1)^{b_1}(-s-z_2)^{b_2}e^{-s}\frac{ds}{s}\\ =&\ \exp(\pi\sqrt{-1}c)\Phi_2(-b_1,-b_2,-c+1;-z_1,-z_2).\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\langle \varphi_2^-, \gamma_1^- \rangle &= -\frac{z_1 \exp(-\pi\sqrt{-1}c)}{(c-1)} \Phi_2(-b_1+1, -b_2, -c+2; -z_1, -z_2), \\
\langle \varphi_3^-, \gamma_1^- \rangle &= -\frac{z_2 \exp(-\pi\sqrt{-1}c)}{(c-1)} \Phi_2(-b_1+1, -b_2, -c+2; -z_1, -z_2).
\end{aligned}$$

Since the intersection number of γ_1^+ and γ_1^- is

$$\begin{aligned}
& \frac{1}{\Gamma(1-c)(1-\exp(2\pi\sqrt{-1}c))} \frac{1}{\Gamma(c)(1-\exp(-2\pi\sqrt{-1}c))}(1-\exp(2\pi\sqrt{-1}c)) \\
= \; & \frac{\sin(\pi c)}{\pi(1-\exp(-2\pi\sqrt{-1}c))} = \frac{\exp(\pi\sqrt{-1}c)}{2\pi\sqrt{-1}},
\end{aligned}$$

the twisted period relation yields that

$$\begin{aligned}
& \Phi_2(b_1, b_2, c; z_1, z_2)\Phi_2(-b_1, -b_2, -c+1; -z_1, -z_2) - 1 \\
= \; & \frac{-1}{c(c-1)}(b_1 z_1 \Phi_2(b_1+1, b_2, c+1; z_1, z_2)\Phi_2(-b_1+1, -b_2, -c+2; -z_1, -z_2) \\
& \qquad + b_2 z_2 \Phi_2(b_1, b_2+1, c+1; z_1, z_2)\Phi_2(-b_1, -b_2+1, -c+2; -z_1, -z_2)).
\end{aligned}$$

5.5 A generalization of Φ_2 (n general)

The function $\Phi_2(\mathbf{b}, c; \mathbf{z}) = \Phi_2(b_1, \ldots, b_r, c; z_1, \ldots, z_r)$ defined by the power series

$$\sum_{k_1, \ldots k_r = 0}^{\infty} \frac{(b_1; k_1)\cdots(b_r; k_r)}{(c; k_1 + \cdots + k_r) k_1! \cdots k_r!} z_1^{k_1} \cdots z_r^{k_r},$$

which converges in $\mathbf{C}^r$, is one of the confluent hypergeometric functions derived from Lauricella's hypergeometric function $F_D(a, b_1, \ldots, b_r, c; z_1, \ldots, z_r)$. By following the previous argument, we have

$$\begin{aligned}
& \Phi_2(\mathbf{b}, c; \mathbf{z})\Phi_2(-\mathbf{b}, -c+1; -\mathbf{z}) - 1 \\
= \; & \frac{-1}{c(c-1)} \left(\sum_{\mu=1}^{r} b_\mu z_\mu \Phi_2(\mathbf{b} + \mathbf{e}_\mu, c+1; \mathbf{z}) \Phi_2(-\mathbf{b} + \mathbf{e}_\mu, -c+2; -\mathbf{z}) \right),
\end{aligned}$$

where $\mathbf{e}_\mu$ is the μ-th unit vector.

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