Comprehensive Restriction Algorithm for Hypergeometric Systems

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1 Introduction

We denote by D the Weyl algebra

$$\mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle,$$

that is the ring of linear partial differential operators with polynomial coefficients. Let M be a holonomic D-module on the n-dimensional space $\mathbb{C}^n = \{x = (x_1, \ldots, x_n)\}$. The 0-th restriction of M to $V(x_{m+1}, \ldots, x_n)$ is defined as

$$\frac{M}{x_{m+1}M + \cdots + x_n M}$$

(see, e.g., [3], [11, Chap 5]). An algorithm computing the restriction was given by T.Oaku [7]. In this paper, we consider a problem of computing the restriction for a given holonomic D-module with parameters. We will give a partial answer to the problem for general holonomic D-modules and an answer to hypergeometric holonomic D-modules.

The basic method for performing various calculations on ideals or submodules of free modules involving parameters is the comprehensive Grb̈ner basis introduced by V.Weispfenning [16]. K.Nabeshima, K.Ohara, S.Tajima [4] introduced comprehesive Gröbner systems (CGS) for rings of linear partial differential operators. They applied their method of computing CGS to the problem of computing b-functions for polynomials with parameters. The parameter space is stratified so that a b-function is associated to each stratum.

For a given holonomic D-module with parameters, we want to stratify the parameter space so that a restriction module that does not depend on parameters is associated to each stratum. We start with generalizing the method by K.Nabeshita et al. to compute a generic b-function that is also called an indicial polynomial or a b-function for restriction (Section 2). The maximal integral root of it plays the central role to apply the Oaku's b-function criterion of the restriction algorithm [7]. The parameter space can be stratified so that a generic b-function is associated to each stratum. However, the difficulty is that roots of it still depends on parameters. We use isomorphic correspondences of

D-modules with parameters to address this difficulty. Since we only need to consider integral roots to obtain the restriction, isomorphic correspondences are fully available.

We focus on algorithms to construct isomorphisms among hypergeometric D-modules in sections 3, 4, 5, 6. M.Saito gave an algorithm to classify GKZ hypergeometric systems into isomorphic classes [10]. We give an algorithm to classify a class of hypergeometric systems of Horn type [2] into isomorphic classes. The key ingradient of our method is constructing strata so that a contiguity relation of a hypergeometric system with parameters is associated to each stratum. Note that a general algorithm to check if two holonomic D-modules are isomorphic or not is given by H.Tsai and U.Walther [15]. Considering a comprehensive version of this algorithm is a future problem.

Utilizing algorithms to classifying isomorphic classes of hypergeometric systems, we finally give a comprehensive restriction algorithm in section 7. The remaining sections 8, 9 are discussions on restrictions to the origin of the Gauss hypergeometric system and the Appell F_1 system.

2 Comprehensive Gröbner System and Generic b-function

K.Nabeshima, K.Ohara, S.Tajima introduced an algorithm for computing comprehensive Gröbner systems (CGS) in rings of linear partial differential operators [4]. They also gave applications of CGS for computing b-functions for singularities. We apply their algorithm to obtain b-functions for weight vectors to compute restrictions of D-modules. See, e.g., [11, Chap 5] on b-functions for weight vectors. Being inspired by the computer algebra system Risa/Asir[1] command name `generic_bfct`, we call them *generic b-functions*. We also call a generic b-function *a b-function for restriction* in this paper to distinguish with a b-function of a contiguity relation and a b-function for a polynomial.

Let

$$D_n[\beta] = \mathbb{C}[\beta_1, \ldots, \beta_m]\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$$

be the Weyl algebra with parameters $\beta = (\beta_1, \ldots, \beta_m)$ regarded as indeterminates. We denote D_n by D when the number of variables is clear. For a left ideal I generated by a set of generators P in $D[\beta]$, we compute a Gröbner basis G with a block order $\succ_b$ satisfying $x_i, \partial_i \succ_b \beta_j$ for any i and j where $\succ$ is a tie-breaker of the block order. Put $E = G \cap \mathbb{C}[\beta]$ and $G' = G \setminus E$. The set $V(E)$ is a given set of equality constraints on parameters. We denote by $\mathrm{CGS}(E, N, P, \succ_b)$ or by $\mathrm{CGS}(E, N, I, \succ_b)$ the output CGS of I on $V(E) \setminus V(N)$ where N is a given set of equality constraints on parameters. The CGS is a finite set of data of the form $(V(E_i) \setminus V(N_i), \mathcal{G}_i)$ where $E_i, N_i \subset \mathbb{C}[\beta]$, $\mathcal{G}_i \subset D[\beta]$ and they are finite set. The CGS has a property that for any $a \in V(E_i) \setminus V(N_i)$, $\mathcal{G}_i|_{\beta=a}$ is a Gröbner basis of $P|_{\beta=a}$ in D with respect to the order $\prec$. $V(E_i) \setminus V(N_i)$ is called a stratum and the strata of this form in the CGS cover $V(E) \setminus V(N)$. Note that

[1] `https://www.openxm.org`

when $E = \emptyset$, we regard $V(E) = \mathbb{C}^m$. The procedure CGS is recursively called. At the top level, we usually start with CGS($E = \emptyset, N = \{1\}, P, \succ_b$). Note that $V(\{1\}) = \emptyset$.

Let w be a vector in $\mathbb{Z}^n_{\geq 0}$. The $(-w, w)$-degree of $ax^p\partial^q$ is $-w \cdot p + w \cdot q$ where $a \in \mathbb{C}[\beta_1, \ldots, \beta_m]$, $a \neq 0$, and $x^p = \prod_{i=1}^n x_i^{p_i}$, $\partial^q = \prod_{i=1}^n \partial_i^{q_i}$. The $(-w, w)$-initial term for $\ell = \sum_{(p,q)\in E} a_{pq}x^p\partial^q$ is the sum of the maximal $(-w, w)$-degree terms of ℓ and is denoted by $\mathrm{in}_{(-w,w)}(\ell)$. For a given left ideal I in $D[\beta]$, the ideal generated by $\mathrm{in}_{(-w,w)}(\ell)$, $\ell \in I$ is called *the initial form ideal* (with respect to the weight vector $(-w, w)$.

Let $\succ$ is a term order in $D[\beta]$. The order $\succ_{(-w,w)}$ is defined as

$$\begin{aligned} & c_{pq}(\beta)x^p\partial^q \succ_{(-w,w)} c'_{p'q'}(\beta)x^{p'}\partial^{q'} \\ \Leftrightarrow\quad & -w \cdot p + w \cdot q > -w \cdot p' + w \cdot q' \\ & \quad \text{or} \ \ (-w \cdot p + w \cdot q = -w \cdot p' + w \cdot q' \ \ \text{and} \ \ x^p\partial^q \succ x^{p'}\partial^{q'}). \end{aligned}$$

Since the order $\succ_{(-w,w)}$ is not a well-order, we need to utilize the homogenized Weyl algebra to compute Gröbner bases with this order. Their CGS algorithm can also be applied to the homogenized Weyl algebra with parameters (see, e.g., [11, Th. 1.2.6] on the homogenized Weyl algebra), and obtain a CGS for the initial form ideal of a given left ideal in $D[\beta]$. This method utilizing the homogenized Weyl algebra is not explicitly described in the paper [4], so we explain it below. Note that the case of the holonomic D-module M is of the form D^m/I where I is a submodule can be discussed analogously.

Algorithm 1 (Computing parametric initial form ideal)**.**

- Input : a set of generators of a left ideal I in $D[\beta]$, a weight vector $w \in (\mathbb{Z}_{\geq 0})^n$
- Output : A stratification of the parameter space $\{(E_i, N_i)\}$ and generators of the initial form ideal $\mathrm{in}_{(-w,w)}(I)$ on the stratum $V(E_i) \setminus V(N_i)$.

1.

2. Let $\succ^h_{(-w,w)}$ be an order in the homogenized Weyl algebra defined as

$$\begin{aligned} & c_{pq}(\beta)x^p\partial^q h^r \succ^h_{(-w,w)} c'_{p'q'}(\beta)x^{p'}\partial^{q'}h^{r'} \\ \Leftrightarrow\quad & -w \cdot p + w \cdot q > -w \cdot p' + w \cdot q' \\ & \quad \text{or} \ \ (-w \cdot p + w \cdot q = -w \cdot p' + w \cdot q' \ \ \text{and} \ \ x^p\partial^q h^r \succ x^{p'}\partial^{q'}h^{r'}). \end{aligned}$$

   where the tie-breaker is an elimination order of h. Extending $\succ^h_{(-w,w)}$ to a block order $\succ^h_{b,(-w,w)}$ such that $x_i, \partial_i, h \succ^h_{b,(-w,w)} \beta_j$, we compute a CGS

$$\mathcal{G} = \{(E_i, N_i, \mathcal{G}_i) \mid i = 1, 2, \ldots, m\}$$

   for I^h.

3. Return $\{(\mathcal{G}_i|_{h=1}, E_i, N_i) \mid i = 1, \ldots, m\}$.

Example 1. Our first example is the system of Appell differential operators for $F_1(a, b, b', c; x, y)$, that is

$$x(1-x)\partial_x^2 + y(1-x)\partial_x\partial_y + (c-(a+b+1)x)\partial_x - by\partial_y - ab, \qquad (1)$$
$$y(1-y)\partial_y^2 + x(1-y)\partial_x\partial_y + (c-(a+b'+1)y)\partial_y - b'x\partial_x - ab', \qquad (2)$$
$$(x-y)\partial_x\partial_y - b'\partial_x + b\partial_y. \qquad (3)$$

They annihilate the function F_1. The left ideal I generated by them are holonomic ideal for any value of the parameter vector and D_2/I is a holonomic D_2-module for any specialization of the parameter vector. For the weight vector $(-w, w), w = (1, 1)$, we apply Algorithm 1 to obtain CGS with respect to the order $\succ_{(-w,w)}$. The parametric initial form ideal is generated for any parameter values (a, b, b', c) by

$$(x-y)\partial_x\partial_y + b\partial_y - b'\partial_x,$$
$$-y\partial_x\partial_y - y\partial_y^2 - b'\partial_x + (b-c)\partial_y,$$
$$-x\partial_x^2 + y\partial_y^2 + (b'-c)\partial_x + (-b+c)\partial_y,$$
$$(-xy+y^2)\partial_y^2 - b'x\partial_x + (b-c)x\partial_y + cy\partial_y.$$

Our second example is the system of Appell differential operators for $F_2(a, b, b', c, c'; x, y)$, that is

$$x(1-x)\partial_x^2 - xy\partial_x\partial_y + (c-(a+b+1)x)\partial_x - by\partial_y - ab, \qquad (4)$$
$$y(1-y)\partial_y^2 - xy\partial_x\partial_y + (c'-(a+b'+1)y)\partial_y - b'x\partial_x - ab'. \qquad (5)$$

For the weight vector $(-w, w), w = (1, 1)$, parametric initial form ideal is generated for any parameter values (a, b, b', c) by

$$-x^2y\partial_x^3\partial_y + xy(x-y)\partial_x^2\partial_y^2 + xy^2\partial_x\partial_y^3 - b'x^2\partial_x^3 + (c'x-(a+b'+c+3)y)x\partial_x^2\partial_y +$$
$$((a+b+c'+3)x-cy)x\partial_x\partial_y^2 + by^2\partial_y^3 - (a+c+2)b'x\partial_x^2 + ((a+b+2)c'x-(a+b'+2)cy)\partial_x\partial_y +$$
$$(a+c'+2)by\partial_y^2 - (a+1)b'c\partial_x + (a+1)bc'\partial_y,$$
$$y\partial_y^2 + c'\partial_y,$$
$$x\partial_x^2 + c\partial_x.$$

In the above examples, there is only one stratum.

The third example is the left ideal generated by $ax\partial_x + by\partial_y$ and $x\partial_x + y\partial_y$ where a, b are parameters. When $a - b \neq 0$, the $(-1, -1, 1, 1)$ initial form ideal is generated by $x\partial_x$ and $y\partial_y$. When $a - b = 0$, it is generated by $x\partial_x + y\partial_y$. There are two strata.

The algorithm [11, Th.5.1.6.] for computing the generic b-function for any weight vector of a holonomic D-ideal can be generalized to ideals with parameters in coefficients as follows.

Algorithm 2 (Parametric generic b-function)**.**

- Input : A set of generators P of a holonomic left ideal I in $D[\beta]$, a weight vector $w \in \mathbb{Z}^n_{\geq 0}$
- Output: Stratification and the generic b-function $b(s)$ on each stratum where $\langle b(s) \rangle = \mathrm{in}_{(-w,w)}(I) \cap \mathbb{C}[s]$ $(s = \sum_{i=1}^n w_i\theta_i,\ \theta_i = x_i\partial_i)$.

1. Compute a parametric initial form ideal $\mathrm{in}_{(-w,w)}(I)$. We obtain the initial form ideal generated by $\mathcal{G}_i$ on each stratum (E_i, N_i) $(i = 1, \dots, r)$.
2. $B \leftarrow \emptyset$
3. For each $i = 1, \dots, r$, do

   3.1 For each element ℓ of $\mathcal{G}_i$, make a replacement $x_k \to u_k x_k$, $\partial_k \to v_k\partial_k$ where k runs over a set of indices such that $w_k \neq 0$. Let J_i be the left ideal geneted by these ℓ's and $1 - u_k v_k$.

   3.2 Compute a CGS for the left ideal J_i on the stratum (E_i, N_i). We use an elimination order $\succ$ of u_k, v_k. (call $\mathrm{CGS}(E_i, N_i, J_i, \succ_b)$.) Let $\mathcal{G}_{ij}$ and stratum (E_{ij}, N_{ij}) $(j = 1, \dots, s)$ be the output. Collect all elements that do not contain u_k, v_k from $\mathcal{G}_{ij}$ and put them in $\mathcal{G}'_{ij}$.

   4 For each $j = 1, \dots, s$, do

   4.1 Any element P of $\mathcal{G}'_{ij}$ is of the form $P = x^a p(\theta)\partial^b$. Replace it as $[\theta]^a p(\theta - b)[\theta]_b$ and put J_{ij} the ideal generated by them where $\theta_i = x_i\partial_i$. Here, $[\theta]^a = \prod_{j=1}^n \prod_{l=1}^{a_j}(\theta_j + l)$ and $[\theta]_b = \prod_{j=1}^n \prod_{l=0}^{b_j-1}(\theta_j - l)$, [11, p.45, p.195].

   4.2 Add $s - \sum_{i=1}^n w_i x_i \partial_i$ to the ideal J_{ij}. Regard it as an ideal in $\mathbb{C}[\beta]\langle \theta_1, \dots, \theta_n, s\rangle$, compute a CGS, and obtain the generator of $J_{ij} \cap \mathbb{C}[s]$ on the stratum (E_{ijk}, N_{ijk}). In other words, compute $\mathrm{CGS}(E_{ij}, N_{ij}, J_{ij}, \succ'_b)$ where $\succ'$ is an order satisfying $x, \partial_x \succ' s$ and take the minimal degree polynomial $b(s)$ of s with coefficients in $\mathbb{C}[\beta]$ for each stratum. Add the polynomial $b(s)$ and the stratum to B.

   Return B.

Example 2. The generic b-function for $(-w, w)$, $w = (1, 1)$ of the Appell system of $F_1(a, b, b', c)$ is

$$b(s) = s(s + c - 1)$$

on $\mathbb{C}^4 = \{(a, b, b', c)\}$.

The generic b-functions for $(-w, w)$, $w = (1, 1)$ of the Appell system of $F_2(a, b, b', c, c')$ are

stratum	generic b-function
$V(0) \setminus V((c - c')(c + c' - 2))$	$s(s + c - 1)(s + c' - 1)(s + c + c' - 2)$
$V(c - c')$	$s(s + c' - 1)(s + 2c' - 2)$
$V(c + c' - 2) \setminus V(c - c')$	$s(s - c' + 1)(s + c' - 1)$

3 Review on Algorithms for Contiguity Relations

In this section, we review known algorithms to find contiguity relations. In the sections 4 and 5, we propose new algorithms to find contiguity relations.

Let $\mathbf{K}$ be a rational function field $\mathbb{C}(\beta_1, \ldots, \beta_d)$. Let $D_n = \mathbf{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ be the Weyl algebra of n variables over the field $\mathbf{K}$. We denote D_n by D when the number of variables is clear. We consider a family of holonomic D-modules $M(\beta) = D/I(\beta)$ where $I(\beta)$ is a left ideal of D. The parameters β_i's are specialized to complex numbers in some context.

In what follows in this section, we assume that the parameters are specialized to be numbers. Let $H_i(\beta)$ be an element of D satisfying the condition

$$\ell\, H_i(\beta) \in I(\beta) \text{ for all } \ell \in I(\beta + e_i). \tag{6}$$

Here $\beta + e_i$ means $(\beta_1, \ldots, \beta_{i-1}, \beta_i + 1, \beta_{i+1}, \ldots, \beta_d)$. Then, we have the left D-morphism

$$\varphi_i : M(\beta + e_i) \ni [p] \longrightarrow [pH_i(\beta)] \in M(\beta). \tag{7}$$

The morphism φ_i induces the morphism of vector spaces of the opposite direction

$$\mathrm{Hom}_D(M(\beta), \hat{\mathcal{O}}_a) \ni f \longrightarrow H_i(\beta) \bullet f \in \mathrm{Hom}_D(M(\beta + e_i), \hat{\mathcal{O}}_a) \tag{8}$$

where $\hat{\mathcal{O}}_a$ is a germ of formal power series at a point $x = a$. When the morphism (7) is an isomorphism, the opposite linear map (8) is also an isomorphism. The operator $H_i(\beta)$ is called the *up-step operator* (for the direction i) or the up-step contiguity operator. Analogously, if we have an element $B_i \in D$ satisfying

$$\ell\, B_i(\beta + e_i) \in I(\beta + e_i) \text{ for all } \ell \in I(\beta), \tag{9}$$

then we have a left D-morphism

$$\psi : M(\beta) \ni [p] \longrightarrow [pB_i(\beta + e_i)] \in M(\beta + e_i), \tag{10}$$

the operator $B_i(\beta)$ is called the *down-step operator* or the down-step contiguity operator.

Regard β as indeterminates. We consider the composite

$$M(\beta) \stackrel{B_i(\beta+e_i)H_i(\beta)}{\longrightarrow} M(\beta), \quad M(\beta + e_i) \stackrel{H_i(\beta)B_i(\beta+e_i)}{\longrightarrow} M(\beta + e_i) \tag{11}$$

When they are multiplications of a polynomial in β, it is called *a b-function of the contiguity* among β and $\beta + e_i$. When the value of the b-function is not zero at a value of β, contiguity operators give an isomorphism among $M(\beta)$ and $M(\beta + e_i)$. We call the set of up-step operator, down-step operator, and the b-function *contiguity relation*.

We note that the same name of b-function is also used in the previous section in a different context. If there is a risk of confusions, we call the b-function in the previous section *the b-function for restriction* and the b-function in this section *the b-function of the contiguity.* The letter b is also used to denote a parameter as a traditional way to express parameters of hypergeometric functions. It will not be confusing.

Example 3. We consider the Gauss hypergeometric operator with $a = c, b, c$ and denote it by

$$L(b,c) = x(1-x)\partial_x + (c-(c+b-1)x)\partial_x - cb. \tag{12}$$

Put $\beta_1 = b, \beta_2 = c$, $d = 1$, $I(\beta) = DL(b,c)$ and consider $M(\beta) = D/I(\beta)$. We fix[2] b as a generic complex number and assume also that c is a generic complex number. Put $\theta_x = x\partial_x$. The operator θ_x is called the Euler operator. Since

$$xL(c) = \theta_x(\theta_x + c - 1) - x(\theta_x + c)(\theta_x + b) = (\theta_x + c - 1)(\theta_x - x(\theta_x + b)),$$

the classical solution space of it is spanned by

$$\begin{aligned} f_1(c) &= (1-x)^{-b} \\ f_2(c) &= x^{1-c}{}_2F_1(1, 1+b-c, 2-c; x) \end{aligned}$$

as a vector space over $\mathbb{C}$. An up-step operator and a down-step operator with respect to c are

$$H(c) = (x-1)\partial_x + c \tag{13}$$
$$B(c) = (1-c)\left(x(x-1)\partial_x + bx - c + 1\right). \tag{14}$$

The b-function for the contiguity is

$$c^2(c-b). \tag{15}$$

These operators act to solutions as follows.

$$H(c) \bullet f_1(c) = (c-b)f_1(c+1) \tag{16}$$
$$H(c) \bullet f_2(c) = (c-1)f_2(c+1) \tag{17}$$

and

$$B(c+1) \bullet f_1(c+1) = c^2 f_1(c) \tag{18}$$
$$B(c+1) \bullet f_2(c+1) = c^2 \frac{b-c}{1-c} f_2(c). \tag{19}$$

The operators $H(c)$ and $B(c+1)$ give a left D-isomorphism among $D/I(\beta)$ and $D/I(\beta + e_2)$.

We are interested in the following problem to apply for our comrehensive restriction algorithm;
Problem Find up-step and down-step contiguity operators that give isomorphisms under a restriction of parameter space.

[2]We omit b to represent dependencies on parameters.

Suppose we reparametrize β as $\beta_1 = L_1(\beta'_1, \ldots, \beta'_m), \ldots, \beta_d = L_d(\beta'_1, \ldots, \beta'_m)$ where L_i are a linear forms of β'. We call this reparametrization *a restriction of parameter space*. We regard β' as a new β. For example, $\beta_1 = \cdots = \beta_d = \beta'_1$ is a restriction of parameter space and β'_1 is regarded as a new β. Our problem is to find an up-step operator and a down-step operator, which give an isomorphism, with respect to β'_1.

How do we find these up-step and down-step contiguity operators on a restricted parameter space? There are several methods to find contiguity operators for hypergeometric systems. Here are a list of them.

1. For given an up-step or a down-step operator, deriving an down-step operator or an up-step operator respectively by Gröbner basis [14], [12], [6].
2. Finding contiguity operators by utilizing the middle convolution and some other operators for rigid systems [9].
3. Finding isomorphism among A-hypergeometric systems [10].
4. Finding isomorphism by finding rational solutions of a system of linear differential equations.
5. Finding isomorphism of classical hypergeometric systems by restricting isomorphisms of A-hypergeometric systems.

Each method has advantages and disadvantages. We briefly explain first three known methods by examples. For general description of these method, please refer to the cited papers above. Last two methods are new and we will give general descriptions in next sections together with examples.

3.1 Deriving down(up)-step operator for a given up(down)-step contiguity operator

Suppose that we are given an up-step (resp. a down-step) operator H. The down-step (resp. the up-step) operator can be constructed by a Gröbner basis computation in the ring of differential operators when parameters are generic numbers [14], [6]. Let us explain this method by an example.

Example 4. The Gauss hypergeometric equation in terms of Euler operator is

$$L(a,b,c) \bullet f = 0, \quad L(a,b,c) = \theta_x(\theta_x + c - 1) - x(\theta_x + a)(\theta_x + b). \qquad (20)$$

Put $H_a(a) = \theta_x + a$. By the relation $x(\theta_x + a + 1) = (\theta_x + a)x$ in D, we have

$$L(a+1,b,c)H_a(a) = H_a(a)L(a,b,c) \in DL(a,b,c). \qquad (21)$$

Therefore, the operator $H_a(a)$ is an up-step operator with respect to a. Suppose $H_a(a)$ gives an isomorphism among $D/L(a,b,c)$ and $D/L(a+1,b,c)$. Since the inverse of $H_a(a)$ is a down-step operator $B_a(a+1)$, the relation

$$B_a(a+1)H_a(a) - 1 \equiv 0 \mod DL(a,b,c) \qquad (22)$$

holds. In other words, the down-step operator $B_a(a+1)$ can be obtained by solving the inhomogeneous syzygy equation in D

$$-1 + s_1 H_a(a) + s_2 L(a,b,c) = 0 \tag{23}$$

where s_1, s_2 are unknown elements in D and s_1 is $B_a(a+1)$. There are several algorithms solving inhomogeneous syzygy equations. In this case, computing the Gröbner basis of $(H_a(a), 1), (L(a,b,c), 0)$ in D^2 by the POT order solves the syzygy equation [6]. The Gröbner basis contains an element

$$(a(c-a-1), x(1-x)\partial_x - bx - a + c - 1) = c_1(H_a(a), 1) + c_2(L, 0), \quad c_i \in D,$$

which implies that $B_a(a+1) = \frac{1}{a(c-a-1)} x(1-x)\partial_x - bx - a + c - 1$.

Since classical hypergeometric systems have either a trivial up-step operator or a down-step operator as in (21), we can obtain any contiguity operator for an integral shift for generic values of parameters by a composition and the method of this section.

We call up-step operators $H_i(\beta)$ and down-step operators $B_i(\beta)$ *atomic contiguity operators*. When they give isomorphisms, a composite of them also gives an isomorphism. However, a restriction in the parameter space of the composite does not always give an isomorphism.

Example 5. We denote x_1 by x and D_1 by D. We consider the Gauss hypergeometric system $D/DL(a,b,c)$. The following operators are atomic contiguity operators.

$$\begin{aligned}
H_\alpha(a,b,c) &= x\partial_x + a, && (24)\\
B_\alpha(a,b,c) &= -x(x-1)\partial_x - (bx + a - c), && (25)\\
H_\gamma(a,b,c) &= x(x-1)\partial_x^2 + ((a+b-c+2)x - 1)\partial_x + ((b-c+1)a + (-c+1)b + c^2 - && (26)\\
B_\gamma(a,b,c) &= x\partial_x + c - 1 && (27)
\end{aligned}$$

Although we use β as underminates of the rational function field $\mathbf{K}$ or a parameter vector in a general setting, we use the same symbol β to use the traditional parameter notation of the Gauss function ${}_2F_1$. Since the distinction is clear from the context, we do not think it will cause any confusion.

We compose them as

$$H := H_c(a+1,b,c)H_a(a,b,c), \; B := B_c(a,b,c+1)B_a(a+1,b,c+1).$$

Reducing B by $DL(a+1,b,c+1)$, we obtain

$$\bar{B} = (a-c)(x(x-1)\partial_x + bx - c). \tag{28}$$

It gives an isomorphism among $D/DL(a+1,b,c+1)$ and $D/DL(a,b,c)$ for generic values of parameters. Note that $\bar{B}$ can be divided by $a-c$. When we restrict B to $a=c$, we have $B' = -x^2(x-1)\partial_x^2 - x((b+c+2)x - c - 1)\partial_x - (c+1)(b)x$, which belongs to the left ideal $DL(c+1,b,c+1)$. This means that B' does not give an isomorphism. On the other hand, we can see $D/DL(c+1,b,c+1)$ and $D/DL(c,b,c)$ are isomorphic for generic complex numbers b,c by

$$\bar{H} = \frac{1}{b(b,c)}\left((c-1)(x-1)\partial_x - c(c-1)\right) \tag{29}$$

and by $\bar{B}/(a-c)$ where $b(b,c) = c(c-1)(c-b)$.

This observation shows that a restriction of a composite of atomic contiguity operators, which gives an isomorphism for generic values of parameters, does not always give an isomorphism. However, dividing a factor like $a-c$ might give an isomorphism as we have seen above. Unfortunately we have no proof that this division is always possible.

3.2 Finding contiguity operator for rigid systems

Let us briefly explain a method to construct contiguity relations given in [9, Sec 3.2, Chap 11] by an example. We will construct a contiguity relation with respect to c for the hypergeometric function

$$f(a,b,c;z) = \frac{1}{\Gamma(a+1)}\int_0^1 (z-x)^a x^b (1-x)^c dx.$$

It satisfies the Gauss hypergeometric equation (20) $L(-a,-a-b-c-1,-a-b)\bullet f = 0$. Put $\phi = x^b(1-x)^c$. Put $\phi_+ = (1-x)\phi$. Applying ∂_x to it, we have

$$\partial_x \bullet \phi_+ = (\partial_x - (x\partial_x + 1))\bullet\phi.$$

Now, we apply the fractional derivative $\partial_x^{-\mu}$, $\mu = a+1$ to the both sides. Note that we have the formula

$$\partial_x^{-\mu} x\partial_x = (x\partial_x - \mu)\partial_x^{-\mu}$$

or

$$\partial_x^{-\mu} x\partial_x = \partial_x^{-\mu} x\partial_x\partial_x^{\mu}\partial_x^{-\mu} = \mathrm{Ad}(\partial_x^{-\mu})(x\partial_x)\partial_x^{-\mu}, \quad \mathrm{Ad}(f)L := f^{-1}Lf$$

in the ring of factional differential operators[3] [9, Secs 1.2, 1.3]. Moreover, we have

$$\partial_x^{-\mu}\bullet\varphi(x) := I^{\mu}(\varphi) := \frac{1}{\Gamma(\mu)}\int_c^x \varphi(t)(t-x)^{\mu-1}dt$$

where c is suitably chosen. This action gives a left module structure to the ring of fractional differential operators and a space of holomorphic functions. By utilizing these relations, we have

$$\begin{aligned}
\partial_x^{-\mu}\partial_x\bullet\phi_+ &= \partial_x^{-\mu}(\partial_x - (x\partial_x+1))\bullet\phi && (30)\\
\partial_x\partial_x^{-\mu}\bullet\phi_+ &= (\partial_x - (x\partial_x - \mu + 1))\partial_x^{-\mu}\bullet\phi && (31)\\
\partial_x\bullet I^{\mu}(\phi_+) &= (\partial_x - (x\partial_x - \mu + 1))\bullet I^{\mu}(\phi) && (32)
\end{aligned}$$

[3] We have no rigorous definition of this ring. The term is used as an intuitive wording.

Thus, changing the variable x by z, we have

$$\partial_z \bullet f(a,b,c+1;z) = (\partial_z - (z\partial_z - a)) \bullet f(a,b,c;z) \tag{33}$$

The function $f(a,b,c;z)$ satisfies the ODE

$$L \bullet f(a,b,c;z) = 0, \quad L = \theta_z(\theta_z - a - b - 1) - z(\theta_z - a)(\theta_z - a - b - c - 1) \tag{34}$$

where $\theta_z = z\partial_z$.

There exist differential operators r_3, r_4 such that $r_3\partial - 1 = r_4 L(a,b,c+1) = 0$, because L is irreducible for generic values of a,b,c. In fact, $r_3 = ((z-z^2)\partial_z + (2a+b+c+1)z - a - b)/(a(a+b+c+2))$ and $r_4 = L/(za(a+b+c+2))$. Applying r_3 to (33), we have

$$f(a,b,c+1;z) = r_3(\partial - (z\partial_z - a)) \bullet f(a,b,c;z).$$

Reducing $r_3(\partial_z - (z\partial_z - a))$ by $L(a,b,c)$, we obtain

$$\frac{z(1-z)\partial_z + az + (c+1)}{a+b+c+2} \bullet f(a,b,c;z) = f(a,b,c+1;z) \tag{35}$$

which is a contiguity relation.

Note that when $b = -1$, the operator L is factored as

$$(\theta_z - z(\theta_z - a - c))(\theta_z - a) \tag{36}$$

and then it is not irreducible. However, we are lucky for the case $b = -1$ that the inverse r_3 of ∂_z exists and the method above works for this degenerate case. When $a = 0, b = -1$, there is no inverse of ∂_z modulo L, because the left ideal generated by ∂_z and L is the principal ideal generated by ∂_z. The method of [9] does not give a contiguity relation for this case. Note that a different approach gives the isomorphism. See Example 6, (44), and Section 5. The contiguity derived by methods above agrees with (35) restricted to $a = 0$ and $b = -1$. The agreement seems to be a coincidence. As we have seen in Example 5 it is not always possible to obtain an up-step or a down-step operator by a restriction of parameters.

Finally, we note two things.

It follows from the relation (35) and the comparison of the constant term that the contiguity relation for hypergeometric series $g(a,b,c;z) := F(-a, -a-b-c-1, -a-b; z)$ is

$$\frac{z(1-z)\partial_z + az + (c+1)}{c+1} \bullet g(a,b,c;z) = g(a,b,c+1;z).$$

The Riemann scheme of the ODE $Lf = 0$ is

$$\left\{ \begin{array}{ccc} x=0 & x=1 & x=\infty \\ 0 & 0 & -a \\ a+b+1 & a+c+1 & -a-b-c-1 \end{array} \right\} \tag{37}$$

3.3 Finding isomorphisms among A-hypergeometric systems

Mutsumi Saito [10] gave an algorithm to stratify the parameter space β of a given A-hypergeometric system by isomorphic classes. He also gave an algorithm to construct an isomorphism among isomorphic A-hypergeometric systems with different beta's.

Let us see his construction with an example. Consider

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{38}$$

and a parameter shift

$$\chi = \chi_+ - \chi_-, \ \chi_+ = (1,0,0)^T, \chi_- = (0,1,0). \tag{39}$$

We have $\chi_+ = Au$, $u = (1,0,0,0)^T$ and $\chi_- = Av$, $v = (0,1,0,0)^T$. The monomial ideal M_χ [10, (4.13)] is generated by ∂_1, ∂_3. A heuristic method to find generators of M_χ is an exhaustive search of u satisfying $Au \in \chi + \mathbb{N}A$ until we succeed to find a relevant b-ideal. Then, we can see that the b-ideal B_χ [10, (4,14)] is generated by $b(s) = s_1 + s_3$. We want to construct an operator E such that

$$E\partial^u = b(\beta)\partial^v \bmod H_A(\beta) \tag{40}$$

where we regard β as indeterminates. We may regard E as an inverse operator of ∂^{u-v}. Although [10, Alg 4.2] gives an efficient algorithm to construct E, the following procedure will be easier for small examples. Compute Gröbner basis in the free module in D^2 of $(\partial_1, 1)$, $\{(\ell, 0) \mid \ell \in H_A(\beta)\}$ with the POT order such that $x_1, x_2, x_3, x_4, \partial_1, \partial_3, \partial_4 \succ \partial_2, \beta_1, \beta_2, \beta_3$ [6]. The Gröbner basis contains an element $((\beta_1+\beta_3)\partial_2, x_1\partial_2 + x_3\partial_3)$. Then, we have $E = x_1\partial_2 + x_3\partial_4$. Let $f(\beta; x)$ be a solution of $H_A(\beta)$. Then, we have $E\partial_1 \bullet f(\beta;x) = (\beta_1+\beta_3)\partial_2 \bullet f(\beta;x)$. Since $\partial_i \bullet f(\beta;x) = f(\beta - a_i; x)$ (modulo non-zero constant factor), we have

$$E \bullet f(\beta - e_1; x) = (\beta_1 + \beta_3) f((\beta - e_1) + e_1 - e_2; x). \tag{41}$$

In other words, E gives a up-step operator for $\chi = e_1 - e_2$. Note that E gives an isomorphism of corresponding D-modules under some conditions.

By setting $\beta = (c-1, -a, -b)$, solutions of this A-hypergeometric system can be written by the Gauss hypergeometric function ${}_2F_1(a, b, c; z)$. Assume $a = c$. Then the restriction of E (see Section 5) gives a contiguity for the integer shift of c for ${}_2F_1(c, b, c; z)$.

As to a general construction algorithm of E and b, refer to [10]. Although this method is efficient, a simpler method works for small problems. Let us explain the simple method.

Algorithm 3 (Finding E and b)**.**

- *Input: generators* $\ell_1, \ldots, \ell_m$ *of* $H_A(\beta)$. ∂^u, ∂^v *where* $u, v \in \mathbb{N}_0^d$ *and their supports are disjoint and* $D/H_A(\beta - Au)$ *and* $D/H_A(\beta - Av)$ *are isomorphic.*
- *Output:* $E \in D$ *and* $b \in \mathbb{C}[\beta]$ *such that* $E\partial^u = b\partial^v$ *modulo* $H_A(\beta)$.

1. *Compute a Gröbner basis* G *by the POT order of*

$$\begin{pmatrix} \partial^u \\ 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \begin{pmatrix} \ell_1 \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \begin{pmatrix} \ell_2 \\ 0 \\ 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} \ell_m \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \in D^{m+2} \tag{42}$$

   The tie breaker $\prec$ *of the POT order is* $\beta \prec$ (∂_i*'s in the support of* ∂^v) $\prec$ (*other variables*).

2. *Find an element of the form* $(b\partial^w, c_0, c_1, \ldots, c_m)^T$ *such that* $\partial^w | \partial^v$ *in the Gröbner basis* G.
3. *Put* $E = \partial^{v-w} c_0$ *and return* E *and* b.

Note that each element of β may be degree 1 or 0 polynomials of indeterminates. For example, $\beta = (-c, -c, 1, c-1, c'-1)$ is OK and $\mathbb{C}[\beta]$ means $\mathbb{C}[c, c']$.

The correctness of this algorithm can be shown as follows. The existence of E and b is proved in [10]. Therefore, the Gröbner basis of ∂^u and ℓ_i's must contain an element of the form $\tilde{b}\partial^w$, $\tilde{b} \in \mathbb{C}[\beta]$ whose leading term $\mathrm{in}_\succ(\tilde{b})\partial^w$, divides the leading term $\mathrm{in}_\succ(b)\partial^v$. Note that E and b are not unique in general. Although b and $\tilde{b}$ might be difference polynomials, we denote $\tilde{b}$ by b in the sequel. Since the Gröbner basis is computed by the POT order, we have $b\partial^w = c_0\partial^u + \sum_{i=1}^m c_i \ell_i$ where $c_i \in D$. Applying ∂^{v-w}, we obtain the output.

Let us consider a degenerate case of $\beta = (\beta_1, 0, 1)$ for our A (38). It stands for the case $a = 0, b = -1, c - 1 = \beta_1$, which were considered in Section 3.2. By applying the algorithm of [10], we have

$$U\partial_1 = \beta_1(\beta_1 + 1),\ U = -(x_1x_4 - x_2x_3)\partial_4 + (\beta_1 + 1)x_1, \tag{43}$$

which gives an isomorphism of D modules

$$\begin{aligned} M((\beta_1, 0, 1)) \ni \ell \quad &\longmapsto \quad \ell \frac{U}{\beta_1(\beta_1+1)} \in M((\beta_1 - 1, 0, 1)), \\ M((\beta_1 - 1, 0, 1)) \ni \ell \quad &\longmapsto \quad \ell\partial_1 \in M((\beta_1, 0, 1)) \end{aligned} \tag{44}$$

when $\beta_1(\beta_1 + 1) \neq 0$.

4 Finding contiguity operators by finding rational solutions

Let $\ell_1, \ldots, \ell_m$ are generators of $I(\beta + e_i)$. Then, the condition (6) satisfied by the up-step operator H_i is equivalent to

$$\ell_j \, H_i(\beta) \in I(\beta), \; j = 1, \ldots, m. \tag{45}$$

Let $R_n = \mathbb{C}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$ be the rational Weyl algebra (the ring of differential operators with rational function coefficients). Let $\{s_k \,|\, k = 1, \ldots, r\}$ be a set of the standard monomials with respect to a Gröbner basis G of $I(\beta)$ in R_n. The set is a basis of $R_n/(R_n I(\beta))$ as a vector space over the rational function field $\mathbb{C}(x)$ where $\mathbb{C}(x)$ is an abbreviation of $\mathbb{C}(x_1, \ldots, x_n)$. Then, the operator H_i can be expressed as

$$H_i = \sum_{k=1}^{r} c_k(x) s_k \tag{46}$$

where $c_k(x)$ is an element of $\mathbb{C}(x)$. Reducing $\ell_j H_i$ by the Gröbner basis G, we have $\sum_{k=1}^{r} (L_{ij}^k \bullet c_k) s_k$ where $L_{ij}^k \in R_n$. Then,

$$L_{ij}^k \bullet c_k = 0, \; j = 1, \ldots, m, \; k = 1, \ldots, r \tag{47}$$

should hold since $\ell_j H_i$ belongs to $I(\beta)$. From the above discussion, the problem of finding an up-step operator H_i has been reduced to the problem of finding a rational solution c_k, $k = 1, \ldots, r$ of (47). This system can be transformed into an integrable connection (a Pfaffian system). An algorithm of finding the rational solutions of an integrable connection is given in [1]. We utilize this algorithm to solve (47).

Example 6. Consider the left ideal $I(c)$ generated by $\ell = (\partial_x - (x\partial_x - c))x\partial_x$ in the $D = D_1$ of one variable $x = x_1$. The set of the standard monomials is $\{1, \partial_x\}$ and we set $H = c_0(x) + c_1(x)\partial_x$. From (47), the vector valued function $F = (c_0', c_0, c_1', c_1)^T$ satisfies the equation $\frac{dF}{dx} = PF$ where

$$P = \begin{pmatrix} \frac{cx+1}{x^2-x} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & \frac{1}{x-1} & \frac{(-c+2)x-1}{x^2-x} & \frac{(2c-2)x^2+3x-1}{x^4-2x^3+x^2} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The space of rational functions of this equation is spanned by $(0, -1, \frac{2x-1}{c+1}, \frac{x(x-1)}{c+1})$. Hence, we have

$$H = \frac{x(1-x)}{c+1}\partial_x + 1. \tag{48}$$

By computing the Gröbner basis of the left D submodule generated by $(1, x(1-x)\partial_x + c + 1)$ and $(0, L)$ in D^2 with the POT order, we see that $(-x\partial_x + c +$

$1, (c+1)^2)$ is in the basis. Hence, we have $(-x\partial_x + c + 1)H - (c+1) \in I(c)$, which means that

$$B(c+1) = \frac{-x}{c+1}\partial_x + 1. \tag{49}$$

Remark 1. For D_n-ideals, the set of the standard monomials are not finite. We can apply this method to a finite subset of the set with looking for polynomial solutions instead of rational solutions. If $D_n/H(\beta)$ and $D_n/H(\beta')$ are isomorphic as left D_n-modules, there exists a finite subset $\{s_k\}$ to express $H_i(\beta)$. Hence, if two D_n-modules are isomorphic, the modified method above can find contiguity operators by enlarging the finite subset in finite steps.

5 From contiguity operators of A-hypergeometric systems to those of classical hypergeometric systems

A relation between A-hypergeometric systems and classical hypergeometric systems studied categorically in [2]. We study a relation of them in terms of restriction of D-modules.

We consider an A-hypergeometric ideal $H_A(\beta)$. We assume the $d \times n$ configuration matrix A is of the form $(E_d \mid A')$ where E_d is the $d \times d$ identity matrix. For example,

$$A = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

satisfies this assumption with $A' = (-1, 1, 1)^T$.

Theorem 1. *Assume $A = (E_d \mid A')$.*

1. *The b-function (indicial polynomial) along $x_1 = \cdots = x_d = 1$ is s.*
2. *The restriction*

$$D/((x_1 - 1)D + \cdots (x_d - 1)D) \otimes_D D/H_A(\beta) \tag{50}$$

is isomorphic to

$$\frac{D_{n-d}}{D_{n-d} \cap (H_A(\beta) + (x_1 - 1)D + \cdots (x_d - 1)D)}$$

where $D_{n-d} = \mathbb{C}\langle x_{d+1}, \ldots, x_n, \partial_{d+1}, \ldots, \partial_n\rangle$ and parameters are specialized to complex numbers.

Proof. (1) It follows from the assumption of the form of A, the left ideal $H_A(\beta)$ contains the operator $\theta_i + \sum_{j>m} a_{ij}\theta_j - \beta_i$. We change the variables $x_i \to x_i + 1$, $i = 1, \ldots, d$, then this operator becomes

$$\theta_i + \partial_i + \sum_{j>d} a_{ij}\theta_j - \beta_i.$$

The initial term of this operator with respect to the weight vector $(-w, w) = (-\mathbf{1}_d, \mathbf{0}_{n-d}, \mathbf{1}_d, \mathbf{0}_{n-d})$ is ∂_i where $\mathbf{1}_d$ is a row vector of d ones and $\mathbf{0}_{n-d}$ is a row vector of $n-d$ zeros. Then, the initial ideal with respect to $(-w, w)$ of $H_A(\beta)$ with the new coordinates contains $\partial_1, \ldots, \partial_d$. Therefore

$$\mathbb{C}[\theta_1 + \cdots + \theta_d] \cap \mathrm{in}_{(-w,w)}(H_A(\beta))$$

contains $s = \theta_1 + \cdots + \theta_d$. Since $H_A(\beta)$ is regular holonomic, it is specializable and the b-function is not constant. Thus, we have $b(s) = s$.

The statement (2) follows from the restriction algorithm [7] and (1). □

The Gauss hypergeometric system is the left D_1 module defined by $D_1/(D_1 L)$,

$$L = x(1-x)\partial^2 + (c - (a+b+1)x)\partial - ab \tag{51}$$

where x_1 is denoted by x. The Appell F_1 system is the left D_2 module defined by D_2/I_{F_1} where I_{F_1} is the left ideal generated by (1), (2), (3) where (x_1, x_2) is denoted by (x, y).

The Appell F_2 system is the left D_2 module defined by $D_2/I_{F_2}(a, b, b', c, c')$ where $I_{F_2}(a, b, b', c, c')$ is the left ideal generated by (4), (5).

Theorem 2. *For any parameter value, the restriction of the following A-hypergeometric systems defined by A and β as (52), (53), (54) to $x_1 = \cdots = x_d = 1$ are the Gauss hypergeometric system, the Appell F_1 system, and the Appell F_2 system respectively by changing the variable names appropriately, e.g., (x_6, x_7) is (x, y) in the case of F_2.*

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \beta = (\gamma - 1, -\alpha, -\beta)^T \tag{52}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}, \quad \beta = (-a, -b, -b', c-1)^T \tag{53}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \quad \beta = (-a, -b, -b', c-1, c'-1)^T \tag{54}$$

Proof. Change variables x_i to $x_i + 1$ for $i = 1, \ldots, d$. Compute a $(-w, w)$ Gröbner basis G for the restriction of each A-hypergeometric system with the tie breaking block order satisfying $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \succ \beta_1, \ldots, \beta_d$ (parameters are last). Computation by a computer program shows that $(-w, w)$ order of each element of G is positive, 0, or -1. See 2024-08-09-gkzF1.rr, 2024-08-09-gkzGauss-rest.rr. Note that the b-function for the restriction is s by Theorem 1. Then, the restriction is generated by $g_{x_1 = \cdots = x_d = 0}, \{g \in G \mid \mathrm{ord}_{(-w,w)}(g) = 0\}$

and $(\partial_i g)_{x_1=\cdots=x_d=0}, \{g \in G \,|\, \mathrm{ord}_{(-w,w)}(g) = -1\}, i = 1, \ldots, d$. Computation by a computer program shows that the restriction agrees with the corresponding classical hypergeometric systems. □

5.1 Restriction of a left D-homomorphism

Let $D = D_n$ be the Weyl algebra in n variables and I a left holonomic ideal in D. b-function along $x_n = 0$ is the monic generator $b(\theta_n)$ of the principal ideal $\mathrm{in}_{(-w,w)}(I) \cap \mathbb{C}[\theta_n]$ where $w = (0, \ldots, 0, 1)$ and $\theta_n = x_n \partial_n$. Assume k_0 be the maximal non-negative root of $b(s) = 0$. Let

$$F_{k_0} = \sum_{k=0}^{k_0} D_{n-1} \partial_n^k \tag{55}$$

Then, the restriction algorithm [7] gives a Gröbner basis $G \subset F_{k_0}$ such that $D/(I + x_n D)$ is isomorphic to $F_{k_0}/D_{n-1}G$ as the left D_{n-1} module.

The b-function plays a cruicial role in the restriction algorithm. It follows from the definition of the b function that there exists an operator r such that

$$b(\theta_n) - r \in I, \mathrm{ord}_{(-w,w)}(r) \leq -1$$

The key identity is

$$\partial_n^j b(\theta_n) = b(j)\partial_n^j + (b(\theta_n + j) - b(j))\partial_n^j + \partial_n^j r \mod I. \tag{56}$$

Note that $(b(\theta_n + j) - b(j))\partial_n^j \in x_n D$ and $\mathrm{ord}_{(-w,w)}(\partial_n^j r) \leq j - 1$.

We define the normal form of $f \in D$ in $D/(I + x_n D)$ as follows.

1. Remove all ∂_n^j $(j > k_0)$ and x_n in f by (56) modulo $I + x_n D$. The result $\tilde{f}$ is in F_{k_0}.
2. Compute the normal form $\tilde{f}$ by the Gröbner basis G. We denote the result by $\bar{f}$.

Assume $\ell \in D$ defines a left D_n-morphism among D/I and D/I' by $D/I \ni [f] \mapsto [f\ell] \in D/I'$. Since it is well-defined, we have $I\ell \subset I'$. This morphism induces the left D_{n-1}-morphism

$$D/(I + x_n D) \ni [f] \mapsto [f\ell] \in D/(I' + x_n D) \tag{57}$$

It is well-defined because $(I + x_n D)\ell \subset I' + x_n D$. The maximal integral root of the b-function of I' along $x_n = 0$ is denoted by k_0' and the Gröbner basis obtained by applying the restriction algorithm to I' by G'.

Proposition 1. *Assume $k_0 = k_0' = 0$. Then, the morphism (57) is given by*

$$D_{n-1}/D_{n-1}G \ni [f] \mapsto [f\bar{\ell}] \in D_{n-1}/D_{n-1}G' \tag{58}$$

where the normal form $\bar{\ell}$ is taken in $I' + x_n D$.

Proof. Since $F_0 = D_{n-1}$, f does not contain the variables x_n and ∂_n. Then we have $f\bar{\ell} - f\ell \in x_n D + I'$ from $\bar{\ell} - \ell = x_n c_1 + c_2$, $c_1 \in D$, $c_2 \in I'$. Note that $f\bar{\ell} \in D_{n-1}$. □

We note that these results can be easily generalized to the case of the restriction to $x_m = x_{m+1} = \cdots = x_n = 0$.

5.2 Restriction of isomorphisms of A-hypergeometric systems to those of classical hypergeometric systems — restriction of Saito's isomorphism

We can obtain an isomorphisms among a contiguous family of classical hypergeometric system such as the Gauss hypergeometric system and Appell hypergeometric system F_2 by applying the restriction algorithm to isomorphisms constructed by M.Saito [10] as long as the maximal integral root of the b-function for the restriction is 0. Note that this method works for any degenerated parameters.

The general algorithm of computing the restriction of a homomorphism can be described in a simple form for the GKZ system when $A = (E_d, A')$ and the restriction is that to $x_1 = \cdots = x_d = 1$.

Algorithm 4 (LR(left-right)-reduction)**.**

- *Input: Rules* $\partial_i \to \ell_i \in D$, $i = 1, \ldots, d$. *An element* $\ell \in D$.
- *Output:* $\bar{\ell}$ *such that* $\bar{\ell} = \ell$ *modulo* $I + \sum_{i=1}^{d} x_i D$ *where* I *is the left ideal in* D *generated by* $\partial_i - \ell_i$, $i = 1, \ldots, d$.

Repeat
$\ell \leftarrow \ell|_{x_1=\cdots=x_d=0}$;
Choose a term of the form $t := cx^\alpha \partial^\beta \partial_i$, $c \in K$ *in* ℓ *and rewrite*

$$\ell \leftarrow cx^\alpha \partial^\beta \ell_i + (\ell - t)$$

until (*there is no term divided by* ∂_i, $i = 1, \ldots, d$)
Output ℓ *as* $\bar{\ell}$.

It is easy to see $\bar{\ell}$ satisfies the output condition when the algorithm stops. For the GKZ system with $A = (E_d, A')$, we firstly make the change of variables $J = H_A(\beta)|_{x_i \to x_i+1, i=1,\ldots,d}$ and use the rules

$$\partial_i \to \ell_i,\ \ell_i = -x_i\partial_i - \sum_{i=d+1}^{n} a_{ij} x_j \partial_j + \beta_i, \quad i = 1, \ldots, d. \tag{59}$$

Note that $\partial_i - \ell_i$ belongs to the GKZ ideal J. The LR-reduction choosing t by the lexicographic order $\partial_1 \succ \partial_2 \succ \cdots$ stops for this case because ℓ_i contains only the term -$x_i\partial_i$ and other terms of ℓ_i do not contain ∂_k, $k = 1, \ldots, d$. More precisely, it can be proved as follows. Consider the degree$(cx^p\partial^q\partial_i, \partial_i)$ be the degree of $cx^p\partial^q\partial_i$ with respect to ∂_i. The degrees of all terms in $x^p\partial^q\ell_i|_{x_1=\cdots=x_d=0}$ are strictly smaller than the original degree. We use the lexicographic order $\partial_1 \succ \partial_2 \succ \cdots$ to choose the term t. Then the degree of the leading term decreases strictly in a finte steps. Then, the LR-reduction stops.

Example 7. Let us consider the A-hypergeometric system $H_A(\beta)$ for $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $\beta = (c-1, -a, -b)$. The column vectors of A is denoted by a_1, a_2, a_3, a_4. We denote $D_4/H_A(\beta)$ by $M_A(\beta)$. We will restrict $H_A(\beta)$ to $x_1 = x_2 = x_3 = 1$. In other words, we consider

$$\frac{D_4}{H_A(\beta) + (x_1 - 1)D_4 + (x_2 - 1)D_4 + (x_3 - 1)D_4}. \tag{60}$$

The b-function $b(s)$ along $x_1 = x_2 = x_3 = 1$ is s and then the maximal integral root is 0 for any value of β. When $B(\beta + a_1) = (\beta_1 + 1 + \beta_2)(\beta_1 + 1 + \beta_3)$ is not zero, ∂_1 gives an isomorphism

$$M_A(\beta) \in [f] \mapsto [f\partial_1] \in M_A(\beta + a_1) \tag{61}$$

and the inverse of ∂_1 is

$$U_1 = x_2x_3\partial_4 + x_1x_3\partial_3 + x_1x_2\partial_2 + x_1^2\partial_1 + x_1 \tag{62}$$

divided by $B(\beta + a_1)$. See [12] and [10] as to algorithms. Let us compute the normal form $\bar{U_1}$. To do this, we make the change of variables $x_i \to x_i + 1$ $(i = 1, 2, 3)$ in $H_A(\beta)$ and consider the restriction to $x_i = 0$ $(i = 1, 2, 3)$. The operators U_1 is

$$(x_2+1)(x_3+1)\partial_4+(x_1+1)(x_3+1)\partial_3+(x_1+1)(x_2+1)\partial_2+(x_1+1)^2\partial_1+x_1+1 \tag{63}$$

and first order operators in $H_A(\beta)$ is

$$(x_1 + 1)\partial_1 - x_4\partial_4 - \beta_1, \tag{64}$$
$$(x_2 + 1)\partial_2 + x_4\partial_4 - \beta_2, \tag{65}$$
$$(x_3 + 1)\partial_3 + x_4\partial_4 - \beta_3. \tag{66}$$
$$\tag{67}$$

Then,

$$\partial_1 \to x_4\partial_4 + \beta_1, \tag{68}$$
$$\partial_2 \to -x_4\partial_4 + \beta_2, \tag{69}$$
$$\partial_3 \to -x_4\partial_4 + \beta_3 \tag{70}$$
$$\tag{71}$$

are reduction rules (56) obtained by the b-function. Applying these rules to (63) and remove elements in $x_1D_4 + x_2D_4 + x_3D_4$, we obtain

$$\bar{U_1} = \partial_4 + (-x_4\partial_4 + \beta_3) + (-x_4\partial_4 + \beta_2) + (x_4\partial_4 + \beta_1) + 1 \tag{72}$$

Replacing β_i's by a, b, c, we have the contiguity operator

$$\frac{1}{(c-a)(c-b)}\left((1 - x_4)\partial_4 - a - b + 1\right). \tag{73}$$

Note that $\bar{\partial_1}$ is $x_4\partial_4 + c$.

The isomorphism (61) holds when $a = 0, b = -1$ and $c(c+1) \neq 0$ and then it induces the isomorphism of the restriction.

If two A hypergeometric systems $D/H_A(\beta)$ and $D/H_A(\beta')$ are isomorphic, then the restriction of them are isomorphic.

6 Representatives of isomorphic classes

Let us consider hypergeometric systems of Horn type obtained by restricting GKZ hypergeometric systems for $A = (E_d, A')$ to $x_1 = \cdots = x_d = 1$. If no confusion arises, we also denote this system of Horn type by $H_A(\beta)$. We assume $\beta \in \mathbb{Z}^d$ for simplicity. M.Saito show that isomorphic classes of GKZ hypergeometric systems can be described by a set $E_\tau(\beta)$ [10]. Although the result may give a classification algorithm based on a geometry of polyhedra and an algebra of monomials, we propose different approach. Although our algorithm works well for Gauss hypergeometric system and Appell systems for F_1, F_2, we have not yet proved that our algorithm stops in finite steps. One more disadvantage of our method is that it may output isomorphic objects as different objects. Note that it is not known if an isomorphism among two hypergeometric systems of Horn type implies an isomorphism among associated GKZ systems.

Let $V = V(L_1, \ldots, L_m)$ be an affine space defined by the intersection of the zero sets of (independent) linear polynomials $L_i(s_1, \ldots, s_d)$, $i = 1, \ldots, m$. Suppose that V contains an integral point $S(V)$. Then, there exists a set of vectors $v_j(V)$, $j = 1, \ldots, d-m$, $V \cap \mathbb{Z}^d$ can be expressed as $S(V) + \sum_{j=1}^{d-m} \mathbb{Z} v_j(V)$ (an efficient algorithm to find them is given in [5]). We denote by $H(\beta)$ a hypergeometric system of Horn type or a GKZ hypergeometric system.

Algorithm 5.
`procedure representative_candidates`$(V, H(\beta))$

1. Compute contiguity relations of $H(\beta)$ for a basis $\{v_j(V)\}$ and $S(V)$ standing for the affine subspace V and associated b-polynomials $B(V)$.

2. $\mathcal{A} =$ the arrangement defined by $B(V)$ on V.

3. Pick one interior point for the intersection I of each maximal face of $\mathcal{A}$ and $\mathbb{Z}^d$. Let P be the collection of them.

4. For each codimension 1 face f of $\mathcal{A}$, put V' be the affine hull of f, call $P' =$`representative_candidates`$(V', H(\beta))$, and $P = P \cup P'$.

5. return P

Call $P =$`representative_candidates(`$\mathbb{R}^d$`,`$H(\beta)$`)`. Remove redundant elements from P by contiguity relations, then we obtain finite representatives of (some) isomorphic classes. If we keep contiguity relations and defining inequalities of I in each step, this algorithm also gives isomorphisms among isomorphic $D/H(\beta)$'s.

Remark 2. 1. This algorithm may output isomorphic objects as different objects.

2. If the left D_n-module $D_n/H(\beta)$ and $D_n/H(\beta')$ are isomorphic, then the left modules $R_n/R_nH(\beta)$ and $R_n/R_nH(\beta')$ over the rational Weyl algebra is isomorphic. Then, if there is no rational solution by the method of Section 4, the corresponding two D_n-modules are not isomorphic.

Definition 1. Let M be a set of vectors $\{v_j(V)\,|\,j = 1,\dots,d-m\}$ of $\mathbb{Z}^d$. Consider a set of points F in $\mathbb{Z}^d$. We construct a directed graph on vertices F by adding an edge between $p,q \in F$ when there exists $v_j(V)$ such that $q = p+v_j(M)$. When the graph is connected, we call F is of *mesh type* with respect to M.

Theorem 3. *The output of Algorithm 5 gives all representatives of the isomorphic classes when the sets of points I's in the algorithm are mesh type with respect to* $\{v_j(V)\}$*'s in the algorithm.*

Proof. Let s be a point in I. The point s does not lie in the zero set of $B(V)$. Then, the contiguity relation with respect to $v_j(V)$ gives an isomorphism between $D/H(s)$ and $D/H(s+v_j(V))$. Hence, if I is of mesh type, all points in I are connected by isomorphisms associated to $v_j(V)$'s. □

Example 8. The confluent hypergeometric function

$$ {}_1F_1(a,c;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(1)_k(c)_k} x^k $$

is annihilated by

$$ L(a,c) = x\partial_x^2 + (c-x)\partial_x - a \quad (74) $$

It is obtained by restricting the GKZ hypergeometric system for $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ to $x_1 = x_3 = 1$ and by changing the variable $x_2 \mapsto -x_2$. Put $M(a,c) = D/DL(a,c)$ where $D = D_1$. Set $V = \mathbb{R}^2$ and $v_1(V) = (1,0)$ and $v_2(V) = (0,-1)$. Consider the direction $\pm v_1(V) = (1,0)$. We have

$$ M(a,c) \overset{x\partial_x+a}{\longleftarrow} M(a+1,c) \quad (75) $$

$$ M(a+1,c) \overset{-x\partial_x+x+a-c+1}{\longleftarrow} M(a,c) \quad (76) $$

The composite of these left D-morphisms

$$ M(a,c) \ni \ell \mapsto \ell(-x\partial_x+x+a-c+1) \mapsto \ell(-x\partial_x+x+a-c+1)(x\partial_x+a) \in M(a,c) $$

is

$$ a(a-c+1), \quad (77) $$

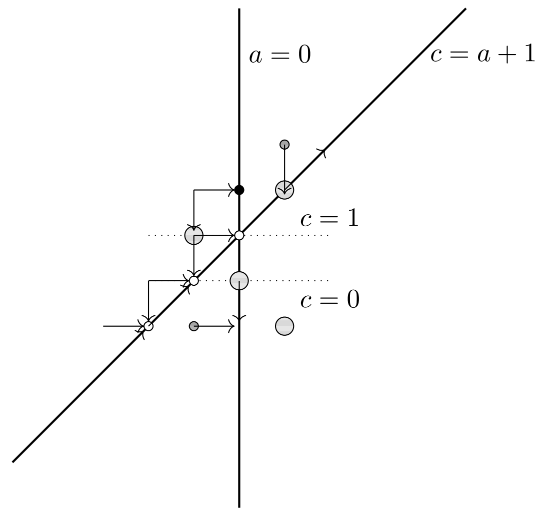

Figure 1: A part of the directed graph and reducing representatives

that is the b-function of this hypergeometric system for the direction $(1,0) \in \mathbb{Z}^2$. Consider the direction $v_2(V) = (0,-1)$. We have

$$M(a,c) \overset{x\partial_x + c - 1}{\longleftarrow} M(a,c-1) \tag{78}$$

$$M(a,c-1) \overset{x\partial_x - 1}{\longleftarrow} M(a,c) \tag{79}$$

and the b-function is

$$a - c + 1. \tag{80}$$

We have four 2-dimensional faces for the arrangement $a(a-c+1) = 0$ in $V = \mathbb{R}^2$.

Secondly, we consider an arrangement on the 1-dimensional space $V = \{a = 0\}$. We have $S(V) = (0,0)$ and $v_1(V) = (0,1)$. The contiguity relation on V is also given by (78) and (79). Then, the arrangement has two 1-dimensional faces and one 0-dimensinal face $(0,1)$.

Finally, we consider arrangement on $V = \{a - c + 1 = 0\}$. We have $S(V) = (0,1)$ and $v_1(V) = (1,1)$. The contiguity relation is given as

$$M(c-1,c) \overset{c\partial_x}{\longleftarrow} M(c,c+1) \tag{81}$$

$$M(c,c+1) \overset{x\partial_x - x + c}{\longleftarrow} M(c-1,c) \tag{82}$$

and the b-function is $c(c-1)$. Then, the arrangement has three 1-dimensional faces and two 0-dimensional faces $(-1,0)$ and $(0,1)$.

All set I obtained by Algorithm 5 for the system for ${}_1F_1$ are of mesh type and Figure 1 illustrates a part of the directed graph of isomorphisms. Big circles of the figure are reduced set of representatives.

Remark 3. Let $\{L_i(s)\}$ be a set of linear polynomials of d-variables with integer coefficients. We consider the arrangment defined by $\{L_i(s) = 0\}$. A problem of finding a Markov basis for the set $\{s \in \mathbb{Z}^n \,|\, L_i(s) > 0 \text{ for all } i\}$ can be reduced to the method of finding a Markov basis for the standard expression of the feasible points $\{u \in \mathbb{N}_0^n \,|\, Au = b\}$ where A is a matrix and b is a vector with integer entries (see, e.g., [13]). We set $s = u - v \in \mathbb{Z}^d$, $u, v \in \mathbb{N}_0^d$ and express the set as $\{(u, v) \in \mathbb{N}_0^{2d} \,|\, L_i(u - v) \geq 1\}$. Adding slack variables, we express the set in the (u, v) space as the standard expression of the feasible points. Thus our reduction is done. A markov basis for the lattice points in a relative interior of a face of the arrangement can be obtained analogously.

Remark 4. The set I can be regarded as feasible points of an integer program. Then, the Markov basis that connects all points in I can be obtained by a Gröbner basis with the trick in the previous remark. If I is not of mesh type, we compute a Markov basis $\{m_j\}$ and contiguity relations for moves $\{\pm m_j\}$. We have new b-functions and the arrangement may become finer. We repeat this procedure until the arrangement does not become finer. Although Saito proved isomorphic classes of $H_A(\beta)$ are finite when $\beta \in \mathbb{Z}^d$ [10], we cannot prove that this repetition stops in finite steps for now. It is a future problem for us to study this method utilizing Markov bases.

Theorem 4. *If $A = (E, *)$ is normal, we can classify the associated Horn systems $H_A(\beta)$ for $\beta \in \mathbb{Z}^d$ into isomorphic classes and compute contiguity relations among isomorphic systems.*

Proof. We denote the GKZ hypergeometric by the same symbol $H_A(\beta)$. Let $F_\sigma(s)$ be the primitive integral supporting function where σ is a facet of the cone generated by the column vectors of A. Since $\beta, \beta' \in \mathbb{Z}^d$, the values of $F_\sigma(\beta)$ and $F_\sigma(\beta')$ belong to $\mathbb{Z}$. Consider the hyperplane arrangement $\mathcal{A}$ defined by $F_\sigma(s) = 0$ where σ runs over the facets. Assume that β and β' belong to the relative interior of a same face of the arrangement. It follows from [10, Th. 5.2] that $M_A(\beta) := D/H_A(\beta)$ and $M_A(\beta')$ are isomorphic, because the theorem says that they are isomorphic if and only if $\beta - \beta' \in \mathbb{Z}A$ and

$$\{\text{facet } \sigma \,|\, F_\sigma(\beta) \in \mathbb{N}_0\} = \{\text{facet } \sigma \,|\, F_\sigma(\beta') \in \mathbb{N}_0\}.$$

Compute a Markov basis for the lattice points in the relative interior of each face of the arrangement $\mathcal{A}$ and contiguity relations associated to the basis. Note that contiguity relations can always be found because $M_A(\beta)$ and $M_A(\beta')$ are isomorphic (Section 4). It follows from Theorem 1, Section 5 and that the isomorphism among $M_A(\beta)$ and $M_A(\beta')$ gives an isomorphism among associated Horn systems that we have completed the proof. □

7 Comprehensive Restriction Algorithm

Let D_n be the Weyl algebra of n variables. Let $I(\kappa)$ be a holonomic left ideal of D_n with parameters $\kappa \in \mathbb{C}^d$. We want to compute the restriction module

$$\frac{D_n}{I(\kappa) + x_1 D_n + \cdots + x_m D_n}. \tag{83}$$

Algorithm 6. (Comprehensive restriction algorithm that gives a partial answer)

- Input: $I(\kappa)$, $x_1 = \cdots = x_m = 0$.
- Output: Strata S_1, S_2 and S_3 of the κ space $\mathbb{C}^d$. The restriction module (83) on each stratum of them.

1. `Ans=[]`.
2. Put $w = (\overbrace{1, \ldots, 1}^{m}, 0, \ldots, 0)$ and compute a comprehensive Gröbner system G by $\prec_{(-w,w)}$ order.
3. Compute comprehensive b-functions for restriction, which are monic generators of $\mathrm{in}_{(-w,w)}(I(\kappa)) \cap \mathbb{C}[\theta_1 + \cdots + \theta_m]$. Let S_1 be strata of the comprehensive b-functions that refines strata of the comprehensive Gröbner system.
4. For each stratum U of S_1, refine U into subsets such that (a) the maximal non-negative integral root of the b-function is 0 or (b) no non-negative integral root of b or (c) other cases on each subset. Let S_2' be the collection of subsets such that (a) or (b) holds. Let S_3' be the collection of subsets such that (c) holds.
5. For each stratum V of S_2', compute a comprehensive Gröbner system of $G'' = G'|_{x_1=\cdots=x_m=0}$ where G' is the collection of the elements of the $(-w, w)$ Gröbner basis G on V such that $\mathrm{ord}_{(-w,w)}(g) \leq 0$, $g \in G$. It refines V and let S_2 be the collection of these refinement.
   (a) The restriction module on each stratum W of S_2 of type (a) is $\frac{D'}{\langle (\text{a Gröbner basis of } G'' \text{ on } W) \rangle}$ where $D' = \mathbb{C}\langle x_{m+1}, \ldots, x_n, \partial_{m+1}, \ldots, \partial_n \rangle$. Append them to `Ans`.
   (b) The restriction module (output) is 0 on the set of strata of S_2' of type (b). Append them to `Ans`.
6. For V in S_3' (type (c)), if $I(\kappa)$, $\kappa \in V$ is a GKZ system or a hypergeometric system of Horn type discussed in previous sections, call $\texttt{Rfr}(I(\kappa(p)), p, G, V)$ (representatives for restriction, Algorithm 7) where we reparametrize κ by p as $\kappa_i(p) = p_i$. The return value is strata S_3 and restrictions on each stratum of S_3. Append them to `Ans`. If $I(\kappa)$ does not belong to these hypergeometric systems, this algorithm does not give an answer.

7. Return `Ans`.

Example 9. Let $\kappa = (a, b, c) \in \mathbb{C}^3$ and consider the left ideal $I(\kappa)$ of D_1 generated by the Gauss hypergeometric operator $L = x(1-x)\partial^2 + (c - (a + b + 1)x)\partial - ab$. Here x_1 is denoted by x and ∂_1 by ∂. The set $\{L\}$ is $(-1, 1)$ Gröbner basis for any κ. Then $S_1 = \{\mathbb{C}^3\}$. We have $\mathrm{in}_{(-1,1)}(L) = x\partial^2 + c\partial$, then the b-function for restriction is $\theta_1(\theta_1 + c - 1)$. Then S_2' is $\{V\}$ where $V = \{(a, b, c) \mid c \notin \mathbb{Z}_{\leq 0}\}$. Since $\mathrm{ord}_{(-1,1)}(L) = 1$, we have $G' = \emptyset$. Then the restriction module on V is $D' = \mathbb{C}$. The strata S_3' (case (c)) is $\{W\}$ where $W = \{(a, b, c) \mid c \in \mathbb{Z}_{\leq 0}\}$. This case will be discussed in Section 8.

Let $\bar{q}_1$ be the maximal non-negative integral root of the b-function for the restriction. Let G be the G that appears in Algorithm 6. We denote by $\texttt{Rest}(G, \bar{q}_1)$ the output of the final step of computing Gröbner basis in a free module of the restriction algorithm, see, e.g., [11, Steps 6 and 7 of Alg. 5.2.8]. Since G contains parameters, the return value is a comprehensive Gröbner system consisting of a strata and Gröbner basis on each stratum.

Example 10. Let $H_2(a, b, b', c, c')$ be the left ideal generated by (4) and (5) that annihilates the Appell function F_2. Consider the left D_2-module $M(a, b, b', c, c') = D_2/H_2(a, b, b', c, c')$. Suppose that $c = 0$ and $c' \notin \mathbb{Z}_{\leq 0}$. The maximum non-negative root of the b-function is $s_0 = 1 - c = 1$ (see Example 2). The restriction module is

$$\frac{\mathbb{C}\partial_x + \mathbb{C}\partial_y + \mathbb{C}}{\mathbb{C}(-ab) + \mathbb{C}c'\partial_y + \mathbb{C}(-ab')}.$$

The dimension is equal to 3 minus the rank of the matrix

$$\begin{pmatrix} 0 & 0 & -ab \\ 0 & c' & -ab' \end{pmatrix}.$$

The stratification with respect to the rank can be obtained by a comprehensive Gröbner system for linear polynomials. $\texttt{Rest}(G, 1)$ returns this comprehensive Gröbner basis.

In order to compute the restriction modules on the strata S_3' for a GKZ system or for a hypergeometric system of Horn type, we apply the following algorithm utilizing algorithms to find contiguity relations. This algorithm is a variation of `representative_candicate` (Algorithm 5).

Algorithm 7. Procedure $\texttt{Rfr}(H_A(\beta(p), p, G', E)$.
Input: $H_A(\beta(p))$ (hypergeometric system), p (a set of m parameters), G' ($(-w, w)$-Gröbner basis), E (conditions).
Output: a list of [conditions(stratum), restriction, contiguity relations].

1. `Ans=[]`.
2. Let $r_1(p), \ldots, r_k(p)$ be the roots of the b-function for the restriction.

3. For all r_i, assume $r_i \in \mathbb{Z}_{\geq 0}$ or not and relationships (larger or smaller or equal) among r_i's supposed to be non-negative integers. Let $\tilde{K}$ be the set of all distinct assumptions on r_i's.
4. For $K \in \tilde{K}$ do
   (a) $q(p)$ be the maximal non-negative integral root under the assumption K. If $q(p)$ is a constant, append $[E \cap K, \texttt{Rest}(G, q(p)), \emptyset]$ to `Ans` and continue the for-loop.
   (b) Changing the indexing, we suppose that $q(p)$ depends on p_1. Introduce new variables $q_1, \ldots, q_m$ such that $q_1 = q = \alpha p_1 + \cdots$ $(\alpha \neq 0)$, $q_i = p_i$ $(i \geq 2)$ and express β by q.
   (c) Choose $\delta \in \mathbb{Z}_{>0}$ so that
   $$\beta_i(q + \delta e_1) - \beta_i(q) \in \mathbb{Z} \quad \text{for all } i \tag{84}$$
   where $e_1 = (1, 0, \ldots, 0)$.
   (d) Put $\Lambda = \{0, 1, 2, \ldots, \delta - 1\}$.
   (e) For k in Λ do
      i. Derive the contiguity relation for it $(L_u, L_d, b(q))$
      $$D/H_A(\beta(q+ke_1)) \overset{L_d}{\to} D/H_A(\beta(q+(k+\delta)e_1)) \overset{L_u}{\to} D/H_A(\beta(q+ke_1)) \tag{85}$$
      where $L_d L_u \equiv b(q)$.
      ii. Consider
      $$(\mathbb{R}^m \setminus V(b)) \cap (\mathbb{R} \times (\bar{q}_2, \ldots, \bar{q}_m)) \tag{86}$$
      where $\bar{q}_2, \ldots, \bar{q}_m$ are generic numbers. Let Q be the set of the $q_1 \in \mathbb{N}_0$ that is the minimum in each first coordinate of connected components of (86).
      iii. Factorize $b(q)$ into degree 1 polynomials as $\prod_{j=1}^{J} b_j(q)$.
      iv. For all $\bar{q}_1 \in Q$ do
         A. Append $[E \cap K \cap \{b(q) \neq 0\}, \texttt{Rest}(G, \bar{q}_1), \text{the contiguity relation}]$.
      v. For all factors b_j in b do
         A. Eliminate one variable in $q_1, \ldots, q_m$ by $b_i(q) = 0$. Changing indices, we suppose that q_m is eliminated.
         B. Express β in terms of $q_1, \ldots, q_{m-1}$.
         C. Append $\texttt{Rfr}(H_A(\beta(q_1, \ldots, q_{m-1}), (q_1, \ldots, q_{m-1}), G, E \cap K \cap \{b_i = 0\})$ to `Ans`.
5. Return `Ans`.

Remark 5. Although, as long as we have tried, we can always find a contiguity relation of δe_1 shift, we might fail at this step. If we fail to find a contiguity relation of δe_1 shift, we need to increase δ. Since the number of isomorphic classes of $H_A(\beta + \iota)$, $\iota \in \mathbb{Z}^d$ are finite by [10], we can find a contiguity relation at a suitable δe_1.

Remark 6. Algorithms 6 and 7 will be generalized to obtain a restriction complex (a restriction of $D_n/I(\kappa)$ in a derived category) by applying the algorithm of [8]. Note that we need to replace "maximal non-negative integral root" of the algorithms 6 and 7 by "maximal integral root". A comprehensive version of $(-w, w)$-adapted resolution is an open question to give an algorithm to obtain a restriction complex.

Example 11. This is a continuation of Example 8 (${}_1F_1$ case). The b-function for the restction to $x = 0$ is $s(s + c - 1)$. The roots are $s = 0$ and $s = 1 - c$. Type (a) case is $1 - c \notin \mathbb{Z}_{\geq 0}$. Since $\mathrm{ord}_{(-1,1)}(L) = 1$, G'' is empty. Then the restriction is $\mathbb{C}$. Since s is a factor of the b-function, type (b) case does not occur.

Consider the type (c) case. In other words, assume $c \in \mathbb{Z}_{\leq 1}$. Let this assumption be E. G is $\{L\}$. We call the procedure $\mathtt{Rfr}(H_A((a, c)), (a, c), G, E)$. Put $q_1 = 1 - c$ and $q_2 = a$. Then, $\Lambda = \{0\}$. The condition on c becomes $q_1 = 1 - c \in \mathbb{Z}_{\geq 1}$ and the roots of the b-function for restriction is 0 and q_1. Firstly, we compute a contiguity relation for the shift from $q_1 = 1-c$ to $q_1+1 = 1 - (c - 1)$. The b-function for contiguity is $q_1 + q_2 = 1 - c + a$ by (80). $\left(\mathbb{R}^2 \setminus V(q_1 + q_2)\right) \cap (\mathbb{R} \times \bar{q}_2)$ is $(-\infty, \infty) \times \bar{q}_2$. Then the set Q of the minimal non-negative integers in the connected component is $\{0\}$. The restriction on this stratum is isomorphic to that of $D/H_A((a, c) = (a, 0))$. Since the maximal integral root is 1 in the (a, c) parameter space and $\mathrm{ord}_{(-1,1)}(L) = 1$, we have $G'' = \{L|_{x=0} = -a\}$. Thus, the restriction is $\mathbb{C}$ when $a \neq 0$ and is $\mathbb{C}^2$ when $a = 0$. Secondly, we consider the case $q_1 + q_2 = 1 - c + a = 0$. The parameter space is one dimensional and parametrized as $(a, c) = (0, 1) - (1, 1)s'$. We call the procedure $\mathtt{Rfr}(H_A((-s', 1 - s')), s', G, \{c \in \mathbb{Z}_{\leq 0}, c = a + 1 = -s' + 1\}$. The b-function for the contiguity of the shift $s' \mapsto s' + 1 \mapsto s'$ is $s'(s' - 1) = c(c - 1)$. Then, the cases of $s' = \{-1, 0, 1\}$ are representatives of isomorphic classes. In other words, $(a, c) = (1, 2), (0, 1), (-1, 0)$ are the representatives. The restrictions are all $\mathbb{C}$.

This procedure will be a little complicated. Then, more examples will help. The comprehensive restriction algorithm will be illustrated for the Gauss hypergeometric system and the system of Appell function F_1 in the following sections 8 and 9.

8 Restriction of the Gauss Hypergeometric System to the Origin

The Gauss hypergeometric function ${}_2F_1(a, b, c; x)$ is annihilated by the operator

$$L(a, b, c) = x(1 - x)\partial_x^2 + (c - (a + b + 1)x)\partial_x - ab. \tag{87}$$

We consider the left ideal generated by L

$$H_g(a, b, c) = DL(a, b, c)$$

where $D = D_1$. We will compute the restriction module

$$M(a,b,c)/xM(a,b,c) \cong D/(H_g(a,b,c) + xD)$$

of the left D-module $M(a,b,c) = D/H_g(a,b,c)$ to $x = 0$.

The generic b-function (for restriction) is

$$b(s) = s(s + c - 1)$$

with respect to the weight vector $w = (1)$. The stratum for this b-function is $\mathbb{C}^3 = \{(a,b,c) \in \mathbb{C}^3\}$. The maximal non-negative integral root s_0 of $b(s)$ is

$$s_0 = \begin{cases} 0 & (c \notin \mathbb{Z}_{\leq 0}) \\ 1 - c & (c \in \mathbb{Z}_{\leq 0}). \end{cases}$$

The Gröbner basis G of H_g by Algorithm 6 is $\{L\}$. The case (b) does not occur and the stratification S_2 of the case (a) consists of only one stratum

$$\{(a,b,c) \mid c \notin \mathbb{Z}_{\leq 0}\}$$

and the restriction module is isomorphic to $\mathbb{C}$, because $\mathrm{ord}_{(-w,w)}(L) = 1$ and then $G'' = \emptyset$.

Before illustrating steps of the procedure $\texttt{Rfr}(H_g, (a,b,c), G, c \in \mathbb{Z}_{\leq 0})$, we show a conclusion that is a list of the restrictions depending on c.

(1) When $c \notin \mathbb{Z}_{\leq 0}$, the restriction module is isomorphic to $\mathbb{C}$.

(2) Suppose that $c = 0$. We have $s_0 = 1$ and then $\mathcal{B}_1 = \{1, \partial_x\}$ (see, e.g., [11, Alg. 5.2.8]). Consider the $\mathbb{C}$-vector space with a basis $\mathcal{B}_1$ $\mathbb{C}^2 = \mathbb{C}\cdot 1 + \mathbb{C}\cdot\partial_x$. Sorting the terms in $L(a,b,0)$ by $<_{(-1,1)}$, we have

$$L(a,b,0) = x\partial_x^2 - x^2\partial_x^2 + (-a-b-1)x\partial_x - ab.$$

Since the $(-1,1)$-degree of it is 1, the vector space of the denominator of the restriction module is generated by $(L(a,b,0))|_{x=0}$ that is

$$(L(a,b,0))|_{x=0} = -ab.$$

Thus, the restriction module is $\mathbb{C}^2/V$ where $V = \mathbb{C} \cdot ab$. Therefore, we have two cases as

- When $a = 0$ or $b = 0$, $V = \{0\}$ and the restriction module is a 2-dimensinal vector space $\mathbb{C}^2$.
- When $a \neq 0$ and $b \neq 0$, $V = \mathbb{C}$ and the restriction module is a 1-dimensional vector space $\mathbb{C}$.

(3) When $c \in \mathbb{Z}_{<0}$, we can reduce cases of $c \in \mathbb{Z}_{<0}$ to the case of $c = 0$ by utilizing left D-module isomorphism

$$D/H_g(a,b,c) \cong D/H_g(\overline{a}, \overline{b}, c+1)$$

where $\overline{a}, \overline{b}$ are a or $a+1$ and b or $b+1$ respectively. We will prove this fact in Proposition 2

Note that we do not give a stratification of $\{(a,b,c)\,|, c \in \mathbb{Z}_{<0}\}$ in the last claim above. We will discuss on it after the proposition.

Proposition 2. *When* $c \in \mathbb{Z}_{<0}$*,*

$$D/H_g(a,b,c) \cong D/H_g(\overline{a},\overline{b},0)$$

holds where $\overline{a}, \overline{b}$ *are* a *or* $a+1$ *and* b *or* $b+1$ *respectively.*

Proof. We abbreviate the Gauss hypergeometric operator as $L(c) = L(a,b,c)$ and the left ideal generated by L as $H_g(c) = H_g(a,b,c)$.

The down-step operator $B(c)$ with respect to c satisfies

$$\exists P \in D \text{ s.t. } L(c-1)B(c) = PL(c).$$

The operator

$$B(c) = \theta_x + (c-1)$$

satisfies it. The up-step operator $H(c)$ satisfies

$$L(c+1)H(c) = PL(c).$$

The operator

$$H_g(c) = (1-x)\partial_x + (c-a-b)$$

satisfies it.

Composing left D-module homomorphisms

$$\varphi : D/H_g(c+1) \ni [P] \mapsto [P \cdot H(c)] \in D/H_g(c)$$

$$\psi : D/H_g(c) \ni [P] \mapsto [P \cdot B(c+1)] \in D/H_g(c+1),$$

we have

$$\varphi \circ \psi : D/H_g(c) \ni [P] \mapsto [P \cdot B(c+1) \cdot H(c)] \in D/H_g(c)$$

$$B(c+1) \cdot H(c) \equiv (a-c)(b-c) \bmod H_g(c)$$

$$\varphi \circ \psi = (a-c)(b-c)\mathrm{id}.$$

Reversing the order of the composition, we have

$$\psi \circ \varphi : D/H_g(c+1) \ni [P] \mapsto [P \cdot H(c) \cdot B(c+1)] \in D/H_g(c+1)$$

$$H(c) \cdot B(c+1) \equiv (a-c)(b-c) \bmod H_g(c+1)$$

$$\psi \circ \varphi = (a-c)(b-c)\mathrm{id}.$$

Hence, when $(a-c)(b-c) \neq 0$, we have the isomorphism $D/H_g(a,b,c) \cong D/H_g(a,b,c+1)$.

The isomorphism breaks when

$$a - c = 0 \text{ or } b - c = 0.$$

We derive contiguity relations with respect to c for these cases.

(1) When $a - c = 0$, put $a = c$. The up-step and down-step operators for $H_g(c, b, c)$ with respect to c are

$$B(c) = (1-c)(x(x-1)\partial_x + bx - c + 1),$$
$$H(c) = (x-1)\partial_x + c,$$
$$B(c+1) \cdot H(c) \equiv c^2(b-c) \bmod H_g(c+1),$$
$$H(c) \cdot B(c+1) \equiv c^2(b-c) \bmod H_g(c).$$

Hence, when $c^2(b-c) \neq 0$, $D/H_g(c, b, c) \cong D/H_g(c+1, b, c+1)$ holds.

(1-1) When $b - c = 0$, put $b = c$. The up-step and down-step operators for $H_g(c, c, c)$ with respect to c are

$$B(c) = (1-c)(x(x-1)\partial_x + (2c-1)x - c + 1),$$
$$H(c) = \partial_x,$$
$$B(c+1) \cdot H(c) \equiv c^3 \bmod H_g(c+1),$$
$$H(c) \cdot B(c+1) \equiv c^3 \bmod H_g(c).$$

Hence, when $c^3 \neq 0$, $D/H_g(c, c, c) \cong D/H_g(c+1, c+1, c+1)$ holds.

(2) When $b - c = 0$, put $b = c$. The up-step and down-step operators for $H_g(a, c, c)$ with respect to c are

$$B(c) = (1-c)(x(x-1)\partial_x + ax - c + 1),$$
$$H(c) = (x-1)\partial_x + c,$$
$$B(c+1) \cdot H(c) \equiv -c^2(a-c) \bmod H_g(c+1),$$
$$H(c) \cdot B(c+1) \equiv -c^2(a-c) \bmod H_g(c).$$

When $c^2(a-c) \neq 0$, $D/H_g(a, c, c) \cong D/H_g(a, c+1, c+1)$ holds.

(2-1) The case $a - c = 0$ is reduced to the case 1-1.

□

Example 12. Let us illustrate the behavior of $\texttt{Rfr}(H_g(\beta), p = (a, b, c), G, I)$ where $\beta = (a, b, c)$, $G = \{L\}$ and the condition I is $c \in \mathbb{Z}_{\leq 1}$. We retain symbol names of Algorithm 7 and of the proof of Proposition 2. The roots of the b-function for restriction are $r_1 = 0$ and $r_2 = 1 - c$. Under the assumption I, we have $r_2 \geq r_1$. Then we put

$$q_1(p) = 1 - c, q_2(p) = a, q_3(p) = b.$$

We can set $\delta = 1$ and then $\Lambda = \{0\}$. The contiguity relation is $(B(c), L(c+1), (a-c)(b-c))$. The b-function of contiguity $(a-c)(b-c)$ can be written in terms of q as $b(q) = (q_1 + q_2 - 1)(q_1 + q_3 - 1)$. If q_2 and q_3 are generic numbers, there is only one connected component of (86). The first coordinate of it is

$(-\infty, \infty)$. Then, the minimum is 0 which means $c = 1$ ($q_1 = 0$). Hence, the restriction module is $\mathbb{C}$ when $c \in \mathbb{Z}_{\leq 1}$ and $(a-c)(b-c) \neq 0$. Isomorphisms are given by the contiguity relation.

Let us run `Rfr` recursively with fewer parameter degrees of freedom. Let $b_1(q)$ be $q_1 + q_3 - 1$. We eliminate q_3 by $b_1(q) = 0$ ($b = c$) and we call

$$\texttt{Rfr}(H_g(q_2, 1-q_1, 1-q_1), (q_1, q_2), G, q_1 \in \mathbb{Z}_{\geq 0} \text{ and } q_1 + q_3 - 1 = 0).$$

Note that $(q_2, 1-q_1, 1-q_1) = (a, c, c)$. Let us execute this procedure. The roots of b-function for restriction is 0 and q_1. The δ is 1 and $\Lambda = \{0\}$. As we have seen in the proof of Proposition 2, the contiguity relation is

$$\left((x-1)\partial + c, (1-c)\left(x(x-1)\partial + ax - c + 1\right), -c^2(a-c)\right)$$

where $-c^2(a-c) = -(1-q_1)^2(q_1+q_2-1)$. Assume that $q_2 = a := \bar{q}_2$ is a generic number. The connected component of (86) are

$$(-\infty, 1) \times \bar{q}_2, \quad (1, \infty) \times \bar{q}_2.$$

Then, $Q = \{0, 2\}$. When $q_1 = 0$ ($c = 1$), the restriction module for $H_g(a, 1, 1)$ is a representative of the isomorphic class consisting of $q_1 = \{0\}$ and is $\mathbb{C}$. When $q_1 = 2$ ($c = -1$), the restriction module for $H_g(a, -1, -1)$ is a representative of the isomorphic class consisting of $q_1 = \{2, 3, \ldots\}$. Since $s_0 = 2$ in this case, the restriction module is

$$\frac{\mathbb{C} + \mathbb{C}\partial + \mathbb{C}\partial^2}{\langle L|_{x=0}, : \partial L : |_{x=0}\rangle} = \frac{\mathbb{C} + \mathbb{C}\partial + \mathbb{C}\partial^2}{\mathbb{C}\partial} \simeq \mathbb{C}^2$$

where $: \,:$ denotes the normally ordered expression (see, e.g., [11, p.3]). Since the b-function for the restriction is $-c^2(a-c) = -(1-q_1)^2(q_1+q_2-1)$, we need to call recursively `Rfr` for each factor. For example, for the factor $1 - q_1$, we call the procedure for $H_g(a, 0, 0)$. The b-function for this contiguity is $a(a+1)$. Note that the degree of freedom of the parameters decreases when the recursion depth increases.

We believe that these explain how this process works, so we will skip the rest.

9 Restriction of Appell F_1 System to the Origin

Applying methods discussed in previous sections, we obtain the following theorem.

Theorem 5. *The restrictions of the hypergeometric system for the Appell function F_1 to $x = y = 0$ are as follows. They are $\mathbb{C}$-vector spaces.*

- *When $c \notin \mathbb{Z}_{\leq 0}$, it is $\mathbb{C}$.*
- *When $b = 0$ and $b' = 0$, it is $(\mathbb{C} \cdot 1 + \mathbb{C} \cdot \partial_x + \mathbb{C} \cdot \partial_y)$.*

- *When ($b \neq 0$ or $b' \neq 0$) and $a = 0$, it is* $(\mathbb{C}\cdot 1+\mathbb{C}\cdot\partial_x+\mathbb{C}\cdot\partial_y)/(\mathbb{C}\cdot(-b'\partial_x+b\partial_y)$.
- *When ($b \neq 0$ or $b' \neq 0$) and $a \neq 0$, it is* $(\mathbb{C}\cdot\partial_x+\mathbb{C}\cdot\partial_y)/(\mathbb{C}\cdot(-b'\partial_x+b\partial_y))$.

Our proof is analogous to the case of the Gauss hypergeometric system. Several contiguity relations are used. They are obtained by our implementation of our algorithms. Our implementation and details of the proof are published in the internet[4]. The proof is omitted here.

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