Nobuki Takayama, 2019.12.05, et-3, Studies of distributions by HGM.

Toric model (discrete) defined by A, 1.

Switch to θ (common in statistics) to x or p.

Let $A = (a_{ij})$ be a $d \times n$ matrix ($a_{ij} \in \mathbf{Z}$). We denote by $a_j \in \mathbf{Z}^d$ the j-th column vector of A. We assume that the raw span contains $(1, 1, \ldots, 1)$. For $\beta \in \mathbf{N}_0 A = \mathbf{N}_0 a_1 + \cdots + \mathbf{N}_0 a_n$, the polynomial

$$Z_A(\beta; x) = \sum_{Au=\beta, u \in \mathbf{N}_0^d} \frac{x^u}{u!} = \sum_{Au=\beta, u \in \mathbf{N}_0^d} \frac{\prod x_i^{u_i}}{\prod u_i!} \qquad (1)$$

is called the A-hypergeometric polynomial.

$P(U = u) = \frac{x^u}{u!} / Z_A(\beta; x)$ is a probability distributioin on $Au = \beta$ with a parameter vector x.

Goal: Exact numerical evaluation of the polynomial $Z_A(\beta; x)$ and its derivatives.

This problem is fundamental, e.g., for given number $\bar{T}_i$, obtain x s.t. $E[U_i] = x_i \frac{\partial Z}{\partial x_i} / Z = \bar{T}_i$ (MLE).

Toric model (discrete) 1'. When $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, we have

$$Z((u_1, u_3+u_4, u_1+u_3); p) = \frac{p_1^{u_1} p_3^{u_3} p_4^{u_4}}{u_1! u_3! u_4!} \sum_{i=0}^{n} \frac{(-u_1)_i (-u_4)_i}{(u_3+1)_i (1)_i} \left(\frac{p_2 p_3}{p_1 p_4} \right)^i .$$

Put $f(a) = {}_2F_1(a, b, c; y)$ and

$$F(a) = \begin{pmatrix} f(a) \\ y\partial_y f(a) \end{pmatrix}, \quad M(a) = \frac{1}{a-c+1} \begin{pmatrix} by+a-c+1 & y-1 \\ -aby & a(1-y) \end{pmatrix}$$

Then, we have the contiguity relation

$$F(a) = M(a) F(a+1). \qquad (2)$$

Toric model (discrete) 2. Contiguity relations give a fast algorithm to evaluate $Z_A(\beta; p)$ and its derivatives.

$$\partial_i \bullet Z_A(\beta; x) = Z_A(\beta - a_i; x)$$

Step 0, holonomic rank	$\mathrm{vol}(A)^*$
Step 1, diff eq	A-hg diff-diff, GG sys
Step 1, contiguity (Pfaffian eq)	general alg's[†]
Step 2, initial value	easy
Step 3, numerical solver	matrix factorial alg's[‡]

hgm openxm, search for links to references.

[*] Ohara-T (2009), Holonomic rank of A-hypergeometric differential-difference equations

[†] 1. Hibi-Nishiyama-T (2012), Pfaffian Systems of A-Hypergeometric Equations I, Bases of Twisted Cohomology Groups. 2. Ohara-T (2015), Pfaffian Systems of A-Hypergeometric Systems II — Holonomic Gradient Method. (Macaulay type matrix method)

[‡] Tachibana-Goto-Koyama-T (2018)

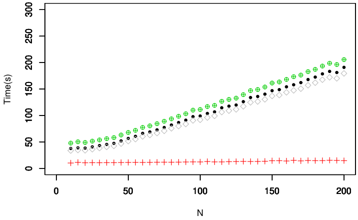
Matrix factorial

We call

$$M(-k)M(-k+1)\cdots M(-2)M(-1)$$

the matrix factorial. Applying the matrix factorial to $F(0)$, we obtain $F(-k)$.

Methods for exact evaluation of matrix factorials (the binary splitting and the modular method)[§].



5×5 contingency table, a benchmark test [tgkt] of evaluating the normalizing constant (A-hypergeometric polynomial) with 32 processes by Risa/Asir with OpenXM. N is a parameter in the marginal sum β.

[§] [tgkt] Y.Tachibana, Y.Goto, T.Koyama, N.Takayama, Holonomic Gradient Method for Two Way Contingency Tables arxiv:1803.04170

Toric model 3, contiguity > series.

Numerical evaluation of hypergeometric polynomial becomes a hard problem when $\dim \mathrm{Ker}\, A$ and the rank of $H_A(\beta)$ increase and β becomes larger.

Example:

$$F_C(a,b,c;y) = \sum_{k \in \mathbf{N}_0^n} \frac{(a)_{|k|}(b)_{|k|}}{\prod k_i! \prod (c_i)_{k_i}} y^k, \quad A = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ E_{n+1} & -E_{n+1} \end{pmatrix}$$

where $(a)_m = a(a+1)\cdots(a+m-1)$ and $|k| = k_1 + \cdots + k_n$.
$n = 4$, $a = -179 - N$, $b = -139 - N$, $c = (37, 23, 13, 31)$,
$y = (31/64, 357/800, 51/320, 87/160)$

N	Evaluating series	method of Macaulay type matrix[¶]
0	6822s (1.89 hour)	61399s (about 17 hours)
100	138640s (1 day and about 14.5 h)	73126s(about 20.3 hours)
200	More than 2 days	84562s (about 23.5 hours)

[¶] Ohara-T (2015)

Contiguity relations give a high quality random vector generator for $Au = \beta$ It is called a *direct sampler*.

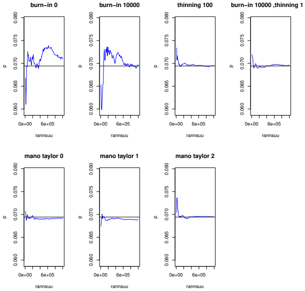
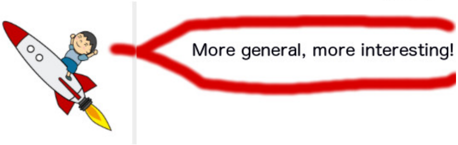


Figure: Evaluation of a p-value by MCMC and direct sampler(Tatuya Hiradai (M2, 2018)). thining 100 is missing in the right.)

1. S.Mano (2016), The A-hypergeometric System Associated with the Rational Normal Curve and Exchangeable Structures. 2. S.Mano (2018), Partitions, Hypergeometric Systems, and Dirichlet Processes in Statistics, JSS Research Series in Statistics.


More special, more interesting!
TK, MY, KO, ...
© dak


NT
Like!
© dak


More general, more interesting!

Contingency table 1

2 way contingency table:
$(k+1)\times(n+1)$ matrix with $\mathbf{Z}_{\geq 0}$ entries.

	acetaminophen	diclofenac sodium	mefenamic acid
death	4	7	2
survival	32	5	6

$$P(U_{ij}=u_{ij})=\frac{\exp(-p_{ij})p_{ij}^{u_{ij}}}{u_{ij}!}$$

The conditional probability ‖ when the row and column sums are fixed to I, J is

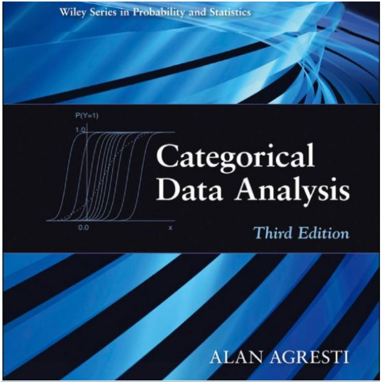
$$P\left(U=u \,\middle|\, \sum_j U_{ij}=I_i, \sum_i U_{ij}=J_j\right)=\frac{p^u/u!}{Z(I,J;p)}$$

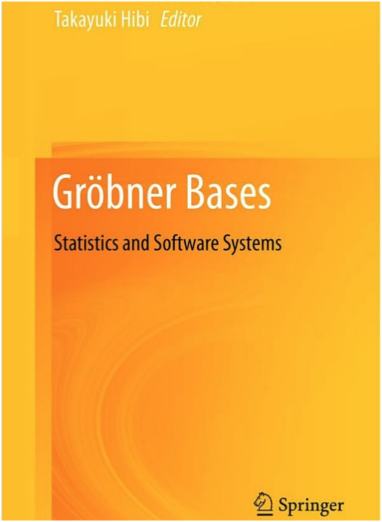
where Z is A-hypergeometric polynomial for $A=(e_i\oplus e_j)$.

Example: $A=\begin{pmatrix}1&1&0&0\\0&0&1&1\\1&0&1&0\\0&1&0&1\end{pmatrix}$ for 2×2 case.

‖ U_{ij} is a random variable of the Poisson distribution.

References on contingency tables (MSC2010: 62H17).





$(k+1) \times (n+1)$ Contingency table

Step 0, holonomic rank	$\binom{k+n}{k}$[**]
Step 1, diff eq	Aomoto-Gel'fand ($\in A$-hg)
Step 1, contiguity (Pfaffian eq)	Goto-Matsutmo (2016)[††]
Step 2, initial value	easy
Step 3, numerical solver	matrix factorial alg's[‡‡]
Packages	`gtt_ekn3.rr` on Risa/Asir

[**] Aomoto (1975, 1977)

[††] Pfaffian equations and contiguity relations of the hypergeometric function of type (k+1,k+n+2) and their applications.

[‡‡] Tachibana-Goto-Koyama-T (2018), Holonomic Gradient Method for Two Way Contingency Tables

Numerical evaluation of Z makes exact MLE possible

Bed time \ Hours slept	less than 6 hour	6–7	more than 7 hours
Before 24	1	6	123
24–25	3	22	145
After 25	86	91	176

Categorical data for males $\begin{pmatrix} 1 & 2 & 28 \\ 0 & 4 & 47 \\ 35 & 32 & 71 \end{pmatrix}$.

Categorical data for females $\begin{pmatrix} 0 & 4 & 95 \\ 3 & 18 & 98 \\ 51 & 59 & 105 \end{pmatrix}$.

CMLE for males: $\begin{pmatrix} 0.458167657900967 & 1 & \underline{6.25676090279981} \\ 0 & 1 & \underline{5.25200491199345} \\ 1 & 1 & 1 \end{pmatrix}$.

CMLE for females:
$$\begin{pmatrix} 0 & 1 & \underline{13.271477373765} \\ 0.193351042187373 & 1 & \underline{3.04872586155291} \\ 1 & 1 & 1 \end{pmatrix}.$$

	acetaminophen	diclofenac sodium	mefenamic acid
death	4	7	2
survival	32	5	6

*

CMLE: $\begin{pmatrix} 1 & \underline{10.5557279737263} & 2.62096714359908 \\ 1 & 1 & 1 \end{pmatrix}$.

Generalized odds ratios: $\begin{pmatrix} 1 & \underline{11.2} & 2.66666666666667 \\ 1 & 1 & 1 \end{pmatrix}$.

$11.2 = \frac{32 \times 7}{4 \times 5}$.

*Data of the previous page https://cran.r-project.org/web/packages/LearnBayes/index.html.

Data of this pagge https://www.pmda.go.jp/files/000148557.pdf

Fisher-Bingham, Bingham distribution 1. PDF:

$\frac{1}{Z(x,y,r)} \exp\left(\sum_{1 \le i \le j \le d+1} x_{ij} t_i t_j + \sum_{i=1}^{d+1} y_i t_i\right) |dt|$ on $S^d(r)$.

$$Z(x,y,r) = \int_{S^d(r)} \exp\left(\sum_{1 \le i \le j \le d+1} x_{ij} t_i t_j + \sum_{i=1}^{d+1} y_i t_i\right) |dt| \qquad (3)$$

$|dt|$ is the invariant measure on the sphere with the redius r such that $\int_{S^d(r)} |dt| = r^d \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}$.

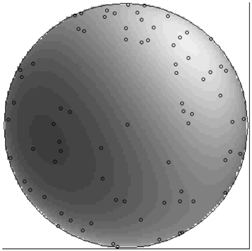
$r = 1, y_i = 0 \Rightarrow$ Bingham distribution *.

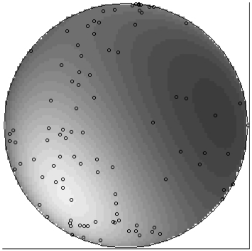
Step 0, holonomic rank	$2d + 2$, Naka-Nishi-Ko-T[yama] (2014)
Step 1, diff eq	Naka-Nishi[yama]-Noro-Ohara-Sei-Takemura-T (2011)
Step 1, Pfaffian eq	Naka-Nishi-Ko-T[yama] (2013). Kume-Sei (2018).
Step 2, initial value	"
Step 3, numerical solver	prototype

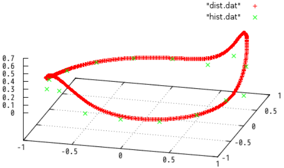
hgm openxm, search for links to references.

*Sei-Kume (2013)

Fisher-Bingham, Bingham distribution 2 (MLE by HGM, stars, wind).







Wind direction at Kobe.

Fisher-Bingham, diff. eq

Put $\partial_{ij} = \partial/\partial x_{ij}$, $\partial_k = \partial/\partial y_k$, $\partial_r = \partial/\partial r$, Z satisfies the following system.

$$\partial_{ij} - \partial_i\partial_j, \quad \sum_{i=1}^{d+1} \partial_i^2 - r^2, \quad r\partial_r - 2\sum_{i\le j} x_{ij}\partial_i\partial_j - \sum_i y_i\partial_i - d,$$

$$x_{ij}\partial_i^2 + 2(x_{jj} - x_{ii})\partial_i\partial_j - x_{ij}\partial_j^2 + \sum_{s\neq i,j}(x_{sj}\partial_i\partial_s - x_{is}\partial_j\partial_s) + y_j\partial_i - y_i\partial_j$$

The normalizing constant of the Bingham distribution satisfies

$$\sum_{i=1}^{d} \partial_{ii} - 1, \quad 2(x_{ii} - x_{jj})\partial_{ii}\partial_{jj} - (\partial_{ii} - \partial_{jj}), \ (1 \le i < j \le d)$$

Fisher distribution $\exp(\mathrm{Tr}\ \Theta^\top X)\delta_{SO(n)}(X)/Z(\Theta)$ on $SO(p) = \{X \in \mathbf{R}^{p\times p} \mid X^T X = I_p, \det(X) = 1\}$ where $\delta_{SO(p)}(X)$ is the delta function supported on $SO(p)$.

$$Z(\Theta) = \int_{X=(X_{ij})\in \mathbf{R}^{p\times p}} \exp(\mathrm{Tr}\ \Theta^\top X)\delta_{SO(p)}(X)dX$$

Step 0, holonomic rank	? (partially [ALSS])
Step 1, diff eq ,	Koyama (2015), any n Any Lie group*
Step 1, Pfaffian eq	$n = 3$†
Step 2, initial value	series expansion ($n = 3$, ")
Step 3, numerical solver	Runge-Kutta method
Packages	`hgm.ncso3` on R ($n = 3$), a prototype for MLE on R($n = 3$) [ALSS]

*[ALSS] M.Adamer, A.Lorincz, A.L.Sattelberger, B.Sturmfels, Algebraic Analysis of Rotation Data, 1912.00396

†Sei-Shibata-Takemura-Ohara-T (2013)

diff eq

$$A_{ij}^{(1)} = \sum_{k=1}^{p} \partial_{ik}\partial_{jk} - \delta_{ij}, \quad \tilde{A}_{ij}^{(1)} = \sum_{k=1}^{p} \partial_{ki}\partial_{kj} - \delta_{ij} \quad (i \le j), \quad A^{(2)} = \det(\partial_{ij}) - 1,$$

$$A_{ij}^{(3)} = \sum_{k=1}^{p} (-\theta_{jk}\partial_{ik} + \theta_{ik}\partial_{jk}), \quad \tilde{A}_{ij}^{(3)} = \sum_{k=1}^{p} (-\theta_{kj}\partial_{ki} + \theta_{ki}\partial_{kj}) \quad (i < j),$$

Here $\partial_{ij} = \partial/\partial\theta_{ij}$.

Note: the diagonal restriction $(\theta_{ij} = 0, i \neq j)$ is not known when $n > 3$.

Wishart matrix (Random matrix) 1 (test) Wishart matrix of the freedom n and $m \times m$ covariance matrix Σ *. Let ℓ_1 is the maximal eigenvalue. Constantine (1963) proved

$$P[\ell_1 < x] = C \exp\left(-\frac{x}{2}\mathrm{Tr}\,\Sigma^{-1}\right) x^{\frac{1}{2}nm} {}_1F_1\left(\frac{m+1}{2}; \frac{n+m+1}{2}; \frac{x}{2}\Sigma^{-1}\right), \tag{4}$$

where C is a constant (omit).

$$\begin{aligned}{}_1F_1(a; c; Y) &= \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \\ &\cdot \int_{0<X<I_m} \exp(\mathrm{Tr}\,XY)|X|^{a-(m+1)/2}|I_m - X|^{c-a-(m+1)/2} dX,\end{aligned}$$

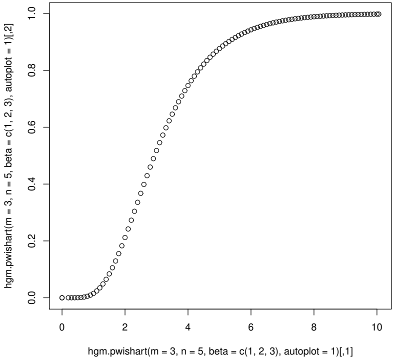
where $\Gamma_m(t) = \pi^{m(m-1)/4}\prod_{i=1}^m \Gamma(t-(i-1)/2)$, $X < Y$ means $Y - X$ is a positive definite symmetric matrix and $dX = \prod_{i \le j} dx_{ij}$.

* M= n-columns of m-dim normal central random variables of the covariance Σ, MM^T is the Wishart matrix.

Wishart matrix 2

Step 0, holonomic rank	2^m
Step 1, diff eq	Muirhead (1970)
Step 1, Pfaffian eq	generic [HNTT] (2013)†, alg by Noro (2016)
Step 2, initial value	...,Koev-Edelman (2005)
Step 3, numerical solver	Runge-Kutta method
Packages	`hgm.cwishart` on R, `n_wishartd.rr`(Noro)

```
library(hgm);
plot(hgm.pwishart(m=3,n=5,
     beta=c(1,2,3),
     autoplot=1))
```



† Hashiguchi-Numata-Takemura-T (2013)

Differential equations (Muirhead, 1970)

$$\left[y_i \partial_i^2 + \left\{ c - \frac{m-1}{2} - y_i + \frac{1}{2} \sum_{j=1, j\neq i}^{m} \frac{y_i}{y_i - y_j} \right\} \partial_i - \frac{1}{2} \sum_{j=1, j\neq i}^{m} \frac{y_j}{y_i - y_j} \partial_j - a \right] F = 0,$$

$$(i = 1, \ldots, m)$$

where $Y = \mathrm{diag}\,(y_1, \ldots, y_m)$. Note: It is zero dimensional in R_m. Holonomic system is not known.

Polyhedron probability of normal distribution for test 1.

Polyhedron parametrized by a_{ij}, b_j

$$P = \{x \in \mathbf{R}^d : \sum_{i=1}^{d} a_{ij} x_i + b_j \geq 0,\ 1 \leq j \leq n\}, \quad (5)$$

$$\mathrm{Prob}(x \in P) = \frac{1}{(2\pi)^{d/2}} \int_{x \in P} \exp\left(-\frac{1}{2}\sum_{i=1}^{d} x_i^2\right) dx_1 \cdots dx_d. \quad (6)$$

Step 0, holonomic rank	$\sharp$(facet intersections$\mathcal{F}$), Koyama (2013)
Step 1, diff eq	Koyama (2013)
Step 1, Pfaffian eq	Koyama (2013)
Step 2, initial value	only simplex case, Koyama (2015)
Step 3, numerical solver	Runge-Kutta method
Packages	prototype when P is a simplex[‡]

[‡] Koyama (2016), http://www.github.com/tkoyama-may10

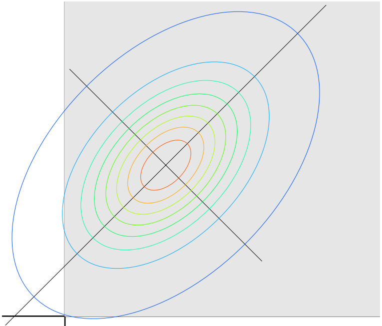
Polyhedron probability of normal distribution for test 2.

$a = [[1,0],[0,1],[-1,-1]]$, $b = [0,0,2]$, in other words, $x > 0, y > 0, -x - y + 2 > 0$ (triangle). $\mathcal{F} = \{\emptyset, 1, 2, 3, 12, 13, 23\}$ and $\sharp\mathcal{F} = 7$.

```
./a.out 1 2 1 0 0 0 1 0 -1 -1 2
result:
   0.1775361552     0.4772498666     0.4772498666
   1.0000000000     0.3100122972     0.1353352805
   0.1353352805
Probability = 0.177536
```

Orthant probability (Koyama, Takemura (2012)):

```
library(hgm);
hgm.ncorthant(
 matrix(c(2,1,1,1),nrow=2),
 y=c(1,1.5));
[1] 0.7475185
```



differential equation (Koyama, 2013)

$$\begin{aligned}
\partial_{a_{ij}} g^J &= \sum_{k=1}^{n} a_{ik} \partial_{b_k} \partial_{b_j} g^J \quad (1 \le i \le d,\ 1 \le j \le n,\ J \in \mathcal{F}), \\
\partial_{b_j} g^J &= g^{J \cup \{j\}} \quad (j \in J^c,\ J \in \mathcal{F}), \\
\partial_{b_j} g^J &= -\sum_{k \in J} \alpha_J^{jk}(a) \left(b_k g^J + \sum_{\ell \in J^c} \alpha_{k\ell}(a) g^{J \cup \{\ell\}} \right) \quad (j \in J,\ J \in \mathcal{F}).
\end{aligned}$$

Here $(\alpha_F^{ij}(a))_{i,j \in F}$ is the ij element of the inverse matrix of $\alpha_F(a)$ constructed by a. ij element of $\alpha_F(a)$ is $\sum_{k=1}^{d} a_{ki} a_{kj}$.

A difficulty of the Step 3, numerical instability [§] Defusing method, stabile subsystem (projection method) [¶].

Example

$$\frac{d}{dt}F = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} F$$

The solution space is spanned by $F^1 = (\exp(-t), 0, 0)^T$, $F^2 = (0, \exp(-t), 0)^T$, $F^3 = (1, 1, 1)^T$. The initial value $(1, 0, 0)^T$ at $t = 0$ yields the solution F^1. Add some errors $(1, 10^{-30}, 10^{-30})^T$ to the initial value. Then, we have

t	value F_1 by RK	difference $F_1 - F^1$
50	1.92827e-22	9.99959e-31
60	8.75556e-27	1.00000e-30
70	1.39737e-30	1.00000e-30
80	1.00002e-30	1.00000e-30

We can see the instability.

[§] I would like to present Some of my recent studies.

[¶] Algorithms to Reduce the Instability of the HGM and Tricks useful for the HGM, preliminary note

$H_n^k(x,y)$: Outage probability 1

$$H_n^k(x,y) = \int_0^x t^k e^{-t} {}_0F_1(\,;n;ty)dt$$

Theorem (Kang-Alouni(2003) ‖)

Under some assumptions, the CDF (cumulative distribution function) of the outage probability (the probability of the maximal eigenvalue $< x$ of a random matrix) is

$$\Pr(\phi_s \leq x) = \frac{\exp(-\sum_{i=1}^s \lambda_i)}{\Gamma(t-s+1)^s \prod_{1 \leq i < j \leq s}(\lambda_j - \lambda_i)} \det \Psi(x)$$

where $\Psi(x)$ is a matrix with $H_{t-s+1}^{t-i}(x, \lambda_j)$ as the (i,j)-element.

They expanded this determinant by the Malcume Q-functions.

Proposition (Danufane-Ohara-T-Siriteanu(2017)**)

The function H_n^k satisfies the following system

$$\begin{aligned} \{\theta_y(\theta_y + n - 1) + y(\theta_x - \theta_y - k - 1)\} \bullet u &= 0, \\ (\theta_x - \theta_y - k - 1 + x)\,\theta_x \bullet u &= 0. \end{aligned}$$

The holonomic rank of the system is 4.

‖ Largest Eigenvalue of Complex Wishart Matrices and Performance Analysis of MIMO MRC Systems

Table: CDF $\mathrm{Prob}(\phi_5 \leq x)$ output by HGM

x	$\mathrm{Pr}(\phi_5 \leq x)$, 5×5	$\mathrm{Pr}(\phi_5 \leq x)$, 5×6	$\mathrm{Pr}(\phi_5 \leq x)$, 5×7
1.9990×10^8	2.841503e-07	2.840696e-07	2.840035e-07
1.9991×10^8	3.372661e-06	3.371785e-06	3.371082e-06
1.9992×10^8	3.147594e-05	3.146851e-05	3.146270e-05
1.9993×10^8	0.00023143703	0.00023138784	0.00023135066
1.9994×10^8	0.0013442040	0.0013439501	0.0013437654
1.9995×10^8	0.0061883532	0.0061873284	0.0061866184
1.9996×10^8	0.022687564	0.022684307	0.022682226
1.9997×10^8	0.066662879	0.066654839	0.066650189
1.9998×10^8	0.15839370	0.15837759	0.15836978
1.9999×10^8	0.30816435	0.30813940	0.30812954
2.0000×10^8	0.49958230	0.49954954	0.49954438
2.0001×10^8	0.69109536	0.69106073	0.69105953
2.0002×10^8	0.84109309	0.84106275	0.84107198
2.0003×10^8	0.93306099	0.93303238	0.93305115
3.0000×10^8	1.000017	0.99999227	1.000260

The rank 4 system is instable. We gave an ODE of rank 3 w.r.t x satisfied by $H_n^k(x, y)$, which is stabile for H_n^k[††].

[††] Th 4 of Danufane-Ohara-T-Siriteanu, Holonomic Gradient Method-Based CDF Evaluation for the Largest Eigenvalue of a Complex Noncentral Wishart Matrix, 2017

Outage probability 3

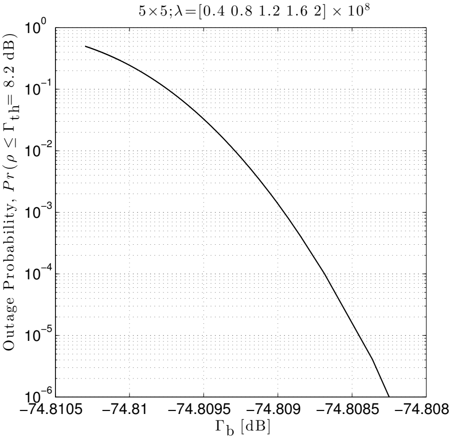
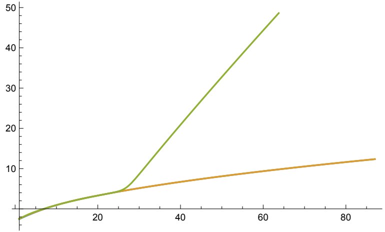
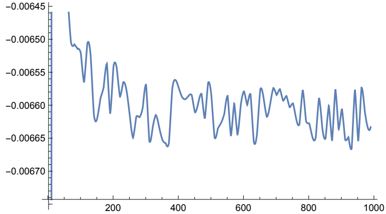


Figure: Outage probability $\Pr(\rho_{\mathsf{MRC}} \leq \Gamma_{\mathsf{th}} = 8.2 \text{ dB}) = \Pr\left(\phi_s \leq \frac{(K+1)\Gamma_{\mathsf{th}}}{\Gamma_{\mathsf{b}}}\right)$ vs. Γ_{b}, from HGM, for $\mathrm{NT} = 5$, $\mathrm{NR} = 5$, and set of eigenvalues $\lambda = \{0.4, 0.8, 1.2, 1.6, 2\} \times 10^8$. Γ_b is the signal noise ratio.

Defusing method for H_n^k



$\log H_1^{10}(1, y)$. Exact value (by numerical integration) and the value by our defusing method agree. The adaptive Runge-Kutta method with the initial relative error 10^{-20} (upper curve) does not agree with the exact value when y is larger than about 25.



The relative error of $H_1^{10}(1, y)$ of our defusing method. The relative error is defined as $(H_d - H)/H$ where H_d is the value by the defusing method and H is the exact value.

A is a random $m \times n$ matrix with the distribution measure $p(A)$. Put $M_x = \{hg^T \mid g^T Ah \geq x, g \in S^{m-1}, h \in S^{n-1}\}$.

Theorem (T-Jiu-Kuriki-Zhang (2018))

The expectation of the Euler characteristic number $E[\chi(M_x)]$ is equal to

$$\frac{1}{2}\int_x^\infty \sigma^{n-m} d\sigma \int_{\mathbb{R}^{(m-1)(n-1)}} dB \int_{S^{m-1}} G^T dg \int_{S^{n-1}} H^T dh \\ \det(\sigma^2 I_{m-1} - BB^T) p(A).$$

We set $G^T dg = \wedge_{i=1}^{m-1} G_i^T dg$, $H^T dh = \wedge_{i=1}^{n-1} H_i^T dh$, where G_i and H_i are the i-th column vectors of G and H, respectively, $dg = (dg_1, \ldots, dg_m)^T$ and $dh = (dh_1, \ldots, dh_n)^T$.

$E[\chi(M_x)]$ approximates $P((\text{max singular value of } A) \geq x)$ when $x \to +\infty$[‡‡]. Problem: Find differential equations for nice $p(A)$'s and apply the HGM.

[‡‡] Adler-Taylor (2007), Kuriki-Takemura (2001)

Summary HGM (holonomic gradient method) has been applied to several holonomic distributions for the (exact) MLE and statistical test in the past 10 years. The method utilizes the thoery of hypergeometric functions, algorithms for the ring of differential operators, and several methods in numerical analysis. The punchline of the method is that it provides a general algorithm to evaluate the normalizing constant and the theoretical study of Z can give more efficient method.