Holonomic rank of $\mathcal{A}$-hypergeometric differential-difference equations

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Abstract

We introduce $\mathcal{A}$-hypergeometric differential-difference equation $\boldsymbol{H}_A$ and prove that its holonomic rank is equal to the normalized volume of $\mathcal{A}$ with giving a set of convergent series solutions.

1 Introduction

In this paper, we introduce $\mathcal{A}$-hypergeometric differential-difference equation $\boldsymbol{H}_A$ and study its series solutions and holonomic rank.

Let $A = (a_{ij})_{i=1,\ldots,d,j=1,\ldots,n}$ be a $d \times n$-matrix whose elements are integers. We suppose that the set of the column vectors of A spans $\mathbf{Z}^d$ and there is no zero column vector. Let a_i be the i-th column vector of the matrix A and $F(\beta, x)$ the integral

$$F(\beta, x) = \int_C \exp\left(\sum_{i=1}^n x_i t^{a_i}\right) t^{-\beta-1} dt, \qquad t = (t_1, \ldots, t_d),\ \beta = (\beta_1, \ldots, \beta_d).$$

The integral $F(\beta, x)$ satisfies the $\mathcal{A}$-hypergeometric differential system associated to A and β "formally". We use the word "formally" because, there is no general and rigorous description about the cycle C ([11, p.222]).

We will regard the parameters β as variables. Then, the function $F(s, x)$ on the (s, x) space satisfies differential-difference equations "formally", which will be our $\mathcal{A}$-hypergeometric differential-difference system.

Rank theories of $\mathcal{A}$-hypergeometric differential system have been developed since Gel'fand, Zelevinsky and Kapranov [4]. In the end of 1980's,

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under the condition that the points lie on a same hyperplane, they proved that the rank of $\mathcal{A}$-hypergeometric differential system $H_A(\beta)$ agrees with the normalized volume of A for any parameter $\beta \in \mathbf{C}^d$ if the toric ideal I_A has the Cohen-Macaulay property. After their result had been gotten, many people have studied on conditions such that the rank equals the normalized volume. In particular, Matusevich, Miller and Walther proved that I_A has the Cohen-Macaulay property if the rank of $H_A(\beta)$ agrees with the normalized volume of A for any $\beta \in \mathbf{C}^d$ ([5]).

In this paper, we will introduce $\mathcal{A}$-hypergeometric differential-difference system, which can be regarded as a generalization of difference equation for the Γ-function, the Beta function, and the Gauss hypergeometric difference equations. As the first step on this differential-difference system, we will prove our main Theorem 3 utilizing theorems on $\mathcal{A}$-hypergeometric differential equations, construction of convergent series solutions with a homogenization technique, uniform convergence of series solutions, and Mutsumi Saito's results for contiguity relations [9], [10], [11, Chapter 4]. The existence theorem 2 on convergent series fundamental set of solutions for $\mathcal{A}$-hypergeometric differential equation for generic β is the second main theorem of our paper. Finally, we note that, for studying our $\mathcal{A}$-hypergeometric differential-difference system, we wrote a program "yang" ([6], [8]) on a computer algebra system Risa/Asir and did several experiments on computers to conjecture and prove our theorems.

2 Holonomic rank

Let $\boldsymbol{D}$ be the ring of differential-difference operators

$$\mathbf{C}\langle x_1, \ldots, x_n, s_1, \ldots, s_d, \partial_1, \ldots, \partial_n, S_1, \ldots, S_d, S_1^{-1}, \ldots, S_d^{-1}\rangle$$

where the following (non-commutative) product rules are assumed

$$S_i s_i = (s_i+1)S_i, \quad S_i^{-1} s_i = (s_i-1)S_i^{-1}, \quad \partial_i x_i = x_i \partial_i + 1$$

and the other types of the product of two generators commute.

Holonomic rank of a system of differential-difference equations will be defined by using the following ring of differential-difference operators with rational function coefficients

$$\mathbf{U} = \mathbf{C}(s_1, \ldots, s_d, x_1, \ldots, x_n)\langle S_1, \ldots, S_d, S_1^{-1}, \ldots, S_d^{-1}, \partial_1, \ldots, \partial_n\rangle$$

It is a $\mathbf{C}$-algebra generated by rational functions in $s_1, \ldots, s_d, x_1, \ldots, x_n$ and differential operators $\partial_1, \ldots, \partial_n$ and difference operators $S_1, \ldots, S_d, S_1^{-1}, \ldots, S_d^{-1}$.

The commutation relations are defined by $\partial_i c(s,x) = c(s,x)\partial_i + \frac{\partial c}{\partial x_i}$, $S_i c(s,x) = c(s_1,\ldots,s_i+1,\ldots,s_d,x)S_i$, $S_i^{-1}c(s,x) = c(s_1,\ldots,s_i-1,\ldots,s_d,x)S_i^{-1}$.

Let I be a left ideal in $\boldsymbol{D}$. The holonomic rank of I is the number

$$\mathrm{rank}(I) = \dim_{\mathbf{C}(s,x)} \mathbf{U}/(\mathbf{U}I).$$

In case of the ring of differential operators ($d = 0$), the definition of the holonomic rank agrees with the standard definition of holonomic rank in the ring of differential operators.

For a given left ideal I, the holonomic rank can be evaluated by a Gröbner basis computation in $\mathbf{U}$.

3 $\mathcal{A}$-hypergeometric differential-difference equations

Let $A = (a_{ij})_{i=1,\ldots,d, j=1,\ldots,n}$ be an integer $d \times n$ matrix of rank d. We assume that the column vectors $\{a_i\}$ of A generates $\mathbf{Z}^d$ and there is no zero vector. The *$\mathcal{A}$-hypergeometric differential-difference system* $\boldsymbol{H}_A$ is the following system of differential-difference equations

$$\begin{aligned} \left(\sum_{j=1}^{n} a_{ij}x_j\partial_j - s_i\right) \bullet f &= 0 \qquad \text{for } i = 1,\ldots,d \quad \text{and} \\ \left(\partial_j - \prod_{i=1}^{n} S_i^{-a_{ij}}\right) \bullet f &= 0 \qquad \text{for } j = 1,\ldots,n. \end{aligned}$$

Note that $\boldsymbol{H}_A$ contains the toric ideal I_A. (use [12, Algorithm 4.5] to prove it.)

Definition 1. Define the unit volume in $\mathbf{R}^d$ as the volume of the unit simplex $\{0, e_1, \ldots, e_d\}$. For a given set of points $\mathcal{A} = \{a_1,\ldots,a_n\}$ in $\mathbf{R}^d$, the normalized volume $\mathrm{vol}(\mathcal{A})$ is the volume of the convex hull of the origin and $\mathcal{A}$.

Theorem 1. *$\mathcal{A}$-hypergeometric differential-difference system $\boldsymbol{H}_A$ has linearly independent* $\mathrm{vol}(A)$ *series solutions.*

The proof of this theorem is divided into two parts. The matrix A is called homogeneous when it contains a row of the form $(1,\ldots,1)$. If A is homogeneous, then the associated toric ideal I_A is homogeneous ideal [12]. The first part is the case that A is homogeneous. The second part is the case that A is not homogeneous.

Proof. (A is homogeneous.) We will prove the theorem with the homogeneity assumption of A. In other words, we suppose that A is written as follows:

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ & * & \end{pmatrix}.$$

Gel'fand, Kapranov, Zelevinski gave a method to construct $m = \mathrm{vol}(A)$ linearly independent solutions of $H_A(\beta)$ with the homogeneity condition of A ([4]). They suppose that β is fixed as a generic $\mathbf{C}$-vector. Let us denote their series solutions by $f_1(\beta; x), \ldots, f_m(\beta; x)$. It is easy to see that the functions $f_i(s; x)$ are solutions of the differential-difference equations $\boldsymbol{H}_A$. We can show, by carefully checking the estimates of their convergence proof, that there exists an open set in the (s, x) space such that $f_i(s; x)$ is locally uniformly convergent with respect to s and x. Let us sketch their proof to see that their series converge as solutions of $\boldsymbol{H}_A$. The discussion is given in [4], but we need to rediscuss it in a suitable form to apply it to the case of inhomogeneous A.

Let B be a matrix of which the set of column vectors is a basis of $\mathrm{Ker}(A : \mathbf{Q}^n \to \mathbf{Q}^d)$ and is normalized as follows:

$$B = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & * & \end{pmatrix} \in M(n, n-d, \mathbf{Q}).$$

We denote by $b^{(i)}$ the i-th column vector of B and by b_{ij} the j-th element of $b^{(i)}$. Then the homogeneity of A implies

$$\sum_{j=1}^{n} b_{ij} = 0.$$

Let us fix a regular triangulation Δ of $\mathcal{A} = \{a_1, \ldots, a_n\}$ following the construction by Gel'fand, Kapranov, Zelevinsky. Take a d-simplex τ in the triangulation Δ. If $\lambda \in \mathbf{C}^n$ is admissible for a d-simplex τ of $\{1, 2, \ldots, n\}$ (*admissible* $\Leftrightarrow$ for all $j \notin \tau$, $\lambda_j \in \mathbf{Z}$), and $A\lambda = s$ holds, then $\boldsymbol{H}_A$ has a formal series solution

$$\phi_\tau(\lambda; x) = \sum_{l \in L} \frac{x^{\lambda + l}}{\Gamma(\lambda + l + 1)},$$

where $L = \mathrm{Ker}(A : \mathbf{Z}^n \to \mathbf{Z}^d)$ and $\Gamma(\lambda + l + 1) = \prod_{i=1}^{n} \Gamma(\lambda_i + l_i + 1)$ and when a factor of the denominator of a term in the sum, we regard the term is zero. Put $\#\tau = n'$. Note that there exists an open set U in the s space such that $\lambda_i,\ i \in \tau$ lie in a compact set in $\mathbf{C}^{n'} \setminus \mathbf{Z}^{n'}$. Moreover, this open set U can be taken as a common open set for all d-simplices in the triangulation Δ and the associated admissible λ's when the integral values λ_j $(j \notin \tau)$ are fixed for all $\tau \in \Delta$.

Put $L' = \{(k_1, \dots, k_{n-d}) \in \mathbf{Z}^{n-d} \mid \sum_{i=1}^{n-d} k_i b^{(i)} \in \mathbf{Z}^n\}$. Then, L' is $\mathbf{Z}$-submodule of $\mathbf{Z}^{n-d}$ and $L = \{\sum_{i=1}^{n-d} k_i b^{(i)} \mid k \in L'\}$. In other words, L can be parametrized with L'. Without loss of the generality, we may suppose that $\tau = \{n-d+1, \dots, n\}$. Then, we have

$$\phi_\tau(\lambda; x) = \sum_{l \in L} \frac{x^{\lambda + l}}{\Gamma(\lambda + l + 1)} = \sum_{k \in L'} \frac{x^{\lambda + \sum_{i=1}^{n-d} k_i b^{(i)}}}{\Gamma(\lambda + \sum_{i=1}^{n-d} k_i b^{(i)} + 1)}$$

Note that the first $n-d$ rows of B are normalized. Then, we have

$$\lambda_j + \sum_{i=1}^{n-d} k_i b_{ij} + 1 = \lambda_j + k_j + 1 \in \mathbf{Z} \qquad (j = 1, \dots, n-d)$$

Since $1/\Gamma(0) = 1/\Gamma(-1) = 1/\Gamma(-2) = \cdots = 0$, the sum can be written as

$$\phi_\tau(\lambda; x) = \sum_{\substack{k \in L' \\ \lambda_j + k_j + 1 \in \mathbf{Z}_{>0} \\ (j=1,\dots,n-d)}} \frac{x^{\lambda + \sum_{i=1}^{n-d} k_i b^{(i)}}}{\Gamma(\lambda + \sum_{i=1}^{n-d} k_i b^{(i)} + 1)}$$

Moreover, when we put

$$\begin{aligned} k'_j &= \lambda_j + k_j, \qquad (j = 1, \dots, n-d) \\ \lambda' &= \lambda - \sum_{i=1}^{n-d} \lambda_i b^{(i)} \\ \hat{\lambda} &= (\lambda_1, \dots, \lambda_{n-d}) \end{aligned}$$

we have

$$\sum_{i=1}^{n-d} k_i b^{(i)} = -\sum_{i=1}^{n-d} \lambda_i b^{(i)} + \sum_{i=1}^{n-d} k'_i b^{(i)}$$

Hence, the sum $\phi_\tau(\lambda; x)$ can be written as

$$\begin{aligned}\phi_\tau(\lambda; x) &= \sum_{\substack{k' \in L' + \hat{\lambda} \\ k' \in \mathbf{Z}^{n-d}_{\geq 0}}} \frac{x^{\lambda - \sum_{i=1}^{n-d} \lambda_i b^{(i)}} \cdot x^{\sum_{i=1}^{n-d} k'_i b^{(i)}}}{\Gamma(\lambda - \sum_{i=1}^{n-d} \lambda_i b^{(i)} + \sum_{i=1}^{n-d} k'_i b^{(i)} + 1)} \\ &= x^{\lambda'} \sum_{\substack{k' \in L' + \hat{\lambda} \\ k' \in \mathbf{Z}^{n-d}_{\geq 0}}} \frac{(x^{b^{(1)}})^{k'_1} \cdots (x^{b^{(n-d)}})^{k'_{n-d}}}{\Gamma(\lambda' + \sum_{i=1}^{n-d} k'_i b^{(i)} + 1)}\end{aligned}$$

Note that our series with the coefficients in terms of Gamma functions agree with those in [11, §3.4], which do not contain Gamma functions, by multiplying suitable constants. Hence we will apply some results on series solutions in [11] to our discussions in the sequel.

Lemma 1. *Let $(k_i) \in (\mathbf{Z}_{\geq 0})^m$ and $(b_{ij}) \in M(m, n, \mathbf{Q})$. Suppose that*

$$\sum_{i=1}^{m} k_i b_{ij} \in \mathbf{Z}, \qquad \sum_{j=1}^{n} b_{ij} = 0$$

and parameters $\lambda = (\lambda_1, \ldots, \lambda_n)$ belongs to a compact set K. Then there exists a positive number r, which is independent of λ, such that the power series

$$\sum_{\substack{k' \in L' + \hat{\lambda} \\ k' \in \mathbf{Z}^{n-d}_{\geq 0}}} \frac{(x^{b^{(1)}})^{k'_1} \cdots (x^{b^{(n-d)}})^{k'_{n-d}}}{\Gamma(\lambda' + \sum_{i=1}^{n-d} k'_i b^{(i)} + 1)}$$

is convergent in $|x^{b^{(1)}}|, \cdots, |x^{b^{(n-d)}}| < r$.

The proof of this lemma can be done by elementary estimates of Γ functions. See [7, pp.18–21] if readers are interested in the details. Since

$$k' \in L' + \hat{\lambda} \Longleftrightarrow \sum_{i=1}^{n-d} k'_i b^{(i)} \in \mathbf{Z}^n$$

it follows from Lemma 1 that there exists a positive constant r such that the series converge in

$$|x^{b^{(1)}}|, \cdots, |x^{b^{(n-d)}}| < r \tag{3.1}$$

for any s in the open set U. We may suppose $r < 1$. Take the log of (3.1). Then we have

$$b^{(k)} \cdot (\log|x_1|, \ldots, \log|x_n|) < \log|r| < 0 \quad \forall k \in \{1, \ldots, n-d\} \tag{3.2}$$

Following [4], for the simplex τ and r, we define the set $C(A,\tau,r)$ as follows.

$$C(A,\tau,r)=\left\{\psi\in\mathbf{R}^n \;\middle|\; \exists\varphi\in\mathbf{R}^d,\;\; \psi_i-(\varphi,a_i)\begin{cases}>-\log|r|, & i\notin\tau,\\ =0, & i\in\tau,\end{cases}\right\}$$

The condition (3.2) and $(-\log|x_1|,\dots,-\log|x_n|)\in C(A,\tau,r)$ is equivalent (see [3, section 4] as to the proof).

Since Δ is a regular triangulation of A, $\bigcap_{\tau\in\Delta}C(A,\tau,r)$ is an open set. Therefore, when s lies in the open set U and $-\log|x|$ lies in the above open set, the $\mathrm{vol}(A)$ linearly independent solutions converge. □

Let us proceed on the proof for the inhomogeneous case. We suppose that A is not homogeneous and has only non-zero column vectors. We define the homogenized matrix as

$$\tilde{A}=\begin{pmatrix}1&\cdots&1&1\\ a_{11}&\cdots&a_{1n}&0\\ \vdots&&\vdots&\vdots\\ a_{d1}&\cdots&a_{dn}&0\end{pmatrix}\in M(d+1,n+1,\mathbf{Z}).$$

For $s=(s_1,\dots,s_n)\in\mathbf{C}^d$ and a generic complex number s_0, we put $\tilde{s}=(s_0,s_1,\dots,s_d)$. We suppose that $\tau=\{n-d+1,\dots,d,d+1\}$ is a $(d+1)$-simplex. Let us take an admissible λ for τ such that $\tilde{A}\tilde{\lambda}=\tilde{s}$ and $\tilde{\lambda}=(\lambda_1,\dots,\lambda_{n+1})\in\mathbf{R}^{n+1}$ as in the proof of the homogeneous case. Put $\lambda=(\lambda_1,\dots,\lambda_n)$. Consider the solution of the hypergeometric system for $\tilde{A}$

$$\tilde{\phi}_\tau(\tilde{\lambda};\tilde{x})=\sum_{k'\in L'\cap S}\frac{\tilde{x}^{\lambda+\sum_{i=1}^{n-d}k'_ib^{(i)}}}{\Gamma(\lambda+\sum_{i=1}^{n-d}k'_ib^{(i)}+1)}$$

and the series

$$\phi_\tau(\lambda;x)=\sum_{k'\in L'\cap S}\frac{\prod_{j=1}^n x_j^{\lambda+\sum_{i=1}^{n-d}k'_ib_{ij}}}{\prod_{j=1}^n\Gamma(\lambda_j+\sum_{i=1}^{n-d}k'_ib_{ij}+1)}$$

($\tilde{x}=(x_1,\dots,x_{n+1})$, $x=(x_1,\dots,x_n)$). Here, the set S is a subset of L' such that an integer in $\mathbf{Z}_{\le 0}$ does not appear in the arguments of the Gamma functions in the denominator. We note that L' for $\tilde{A}$ and L' for A agree, which can be proved as follows. Let $(k_1,\dots,k_{n+1})$ be in the kernel of $\tilde{A}$ in $\mathbf{Q}^{n+1}$. Since $\tilde{A}$ contains the row of the form $(1,\dots,1)$, then $(k_1,\dots,k_n)\in\mathbf{Z}^n$ implies that k_{n+1} is an integer. The conclusion follows from the definition of L'.

Definition 2. We call $\phi_\tau(\lambda; x)$ the *dehomogenization* of $\tilde{\phi}_\tau(\tilde{\lambda}; \tilde{x})$.

Intuitively speaking, the dehomogenization is defined by "forgetting" the last variable x_{n+1} associated Γ factors. See Example 1.

Formal series solutions for the hypergeometric system for inhomogeneous A do not converge in general. However, we can construct vol(A) convergent series solutions as the dehomogenization of a set of series solutions for $\tilde{A}$ hypergeometric system associated to a regular triangulation on $\tilde{\mathcal{A}}$ induced by a "nice" weight vector $\tilde{w}(\varepsilon)$, which we will define. Put $\tilde{w} = (1, \ldots, 1, 0) \in \mathbf{R}^{n+1}$. Since the Gröbner fan for the toric variety $I_{\tilde{A}}$ is a polyhedral fan, the following fact holds.

Lemma 2. *For any $\varepsilon > 0$, there exists $\tilde{v} \in \mathbf{R}^{n+1}$ such that $\tilde{w}(\varepsilon) := \tilde{w} + \varepsilon\tilde{v}$ lies in the interior of a maximal dimensional Gröbner cone of $I_{\tilde{A}}$. We may also suppose $\tilde{v}_{n+1} = 0$.*

Proof. Let us prove the lemma. The first part is a consequence of an elementary property of the fan. When I is a homogeneous ideal in the ring of polynomials of $n + 1$ variables, we have

$$\mathrm{in}_{\tilde{u}}(I) = \mathrm{in}_{\tilde{u}+t(1,\cdots,1)}(I) \tag{3.3}$$

for any t and any weight vector $\tilde{u}$. In other words, $\tilde{u}$ and $\tilde{u} + t(1, \ldots, 1)$ lie in the interior of the same Gröbner cone.

When the weight vector $\tilde{w}(\varepsilon) = \tilde{w} + \varepsilon\tilde{v}$ lies in the interior of the Gröbner cone, we define a new $\tilde{v}$ by $\tilde{v} - \tilde{v}_{n+1}(1, \ldots, 1)$. Since the initial ideal does not change with this change of weight, we may assume that $\tilde{v}_{n+1} = 0$ for the new $\tilde{v}$. □

Since the Gröbner fan is a refinement of the secondary fan and hence $\tilde{w}(\varepsilon)$ is an interior point of a maximal dimensional secondary cone, it induces a regular triangulation ([12] p.71, Proposition 8.15). We denote by Δ the regular triangulation on $\tilde{A}$ induced by $\tilde{w}(\varepsilon)$. For a d-simplex $\tau \in \Delta$, we define $b^{(i)}$ as in the proof of the homogeneous case. Since the weight for $\tilde{a}_{n+1}$ is the lowest, $n + 1 \in \tau$ holds. We can change indices of $\tilde{a}_1, \ldots, \tilde{a}_n$ so that $\tau = \{n - d + 1, \ldots, n + 1\}$ without loss of generality.

Let us prove that the dehomogenized series $\phi_\tau(\lambda; x)$ converge. It follows from a characterization of the support of the series [11, Theorem 3.4.2] that we have

$$\tilde{w}(\varepsilon) \cdot \left(\sum_{i=1}^{n-d} k'_i b^{(i)} + \lambda \right) \geq \tilde{w}(\varepsilon) \cdot \lambda, \qquad \forall k' \in L' \cap S.$$

Here, S is a set such that $\mathbf{Z}_{\leq 0}$ does not appear in the denominator of the Γ factors. Take the limit $\varepsilon \to 0$ and we have

$$\tilde{w} \cdot \sum_{i=1}^{n-d} k'_i b^{(i)} \geq 0, \qquad \forall k' \in L' \cap S.$$

From Lemma 2, $\tilde{w}(\varepsilon) \in C(\tilde{A}, \tau)$ holds and then

$$\tilde{w}(\varepsilon) \cdot b^{(i)} \geq 0.$$

Similarly, by taking the limit $\varepsilon \to 0$, we have

$$\tilde{w} \cdot b^{(i)} = \sum_{j=1}^{n} b_{ij} \geq 0.$$

Therefore, we have $\sum_{j=1}^{n+1} b_{ij} = 0$, the inequality $b_{i,n+1} \leq 0$ holds for all i.

Since $k'_1 \geq -\lambda_1, \dots, k'_{n-d} \geq -\lambda_{n-d}$, we have

$$\sum_{i=1}^{n-d} k'_i b_{i,n+1} \leq -\sum_{i=1}^{n-d} \lambda_i b_{i,n+1}$$

Note that the right hand side is a non-negative number. Suppose that λ_{n+1} is negative. In terms of the Pochhammer symbol we have $\Gamma(\lambda_{n+1} - m) = \Gamma(\lambda_{n+1})(-\lambda_{n+1}+1; m)^{-1}(-1)^m$, then we can estimate the $(n+1)$-th gamma factors as

$$\begin{aligned} \left|\Gamma(\lambda_{n+1} + \sum_{i=1}^{n-d} k'_i b_{i,n+1} + 1)\right| &= |\Gamma(\lambda_{n+1}+1)| \cdot \left|\left(-\lambda_{n+1}; -\sum_{i=1}^{n-d} k'_i b_{i,n+1}\right)\right|^{-1} \\ &\leq c'|\Gamma(\lambda_{n+1}+1)| \cdot \left|\left(-\lambda_{n+1}; -\sum_{i=1}^{n-d} \lambda_i b_{i,n+1}\right)\right|^{-1} \\ &= c \end{aligned} \tag{3.4}$$

Here, c' and c are suitable constants.

When $\lambda_{n+1} \geq 0$, there exists only finite set of values such that $\lambda_{n+1} + \sum_{i=1}^{n-d} k'_i b_{i,n+1} \geq 0$. Then, we can show the inequality (3.4) in an analogous way.

Now, by (3.4), we have

$$\left|\frac{1}{\prod_{j=1}^{n} \Gamma(\lambda_j + \sum_{i=1}^{n-d} k'_i b_{ij} + 1)}\right| \leq c \left|\frac{1}{\Gamma(\lambda + \sum_{i=1}^{n-d} k'_i b^{(i)} + 1)}\right|$$

We note that the right hand side is the coefficient of the series solution for the homogeneous system for $\tilde{A}$ and the series converge for $(-\log|x_1|, \dots, -\log|x_{n+1}|) \in C(\tilde{A}, \tau, r)$ $(r < 1)$ uniformly with respect to $\tilde{s}$ in an open set.

Put $x_{n+1} = 1$. Since $-\log|x_{n+1}| = 0$ and $\tilde{w}(\varepsilon) \in \{y \mid y_{n+1} = 0\}$, we can see that

$$\bigcap_{\tau \in \Delta} C(\tilde{A}, \tau, r) \cap \{y \mid y_{n+1} = 0\}$$

is a non-empty open set of $\mathbf{R}^n$. Therefore the dehomogenized series $\phi_\tau(\lambda; x)$ converge in an open set in the (s, x) space.

Theorem 2. *The dehomogenized series $\phi_\tau(\lambda; x)$ satisfies the hypergeometric differential-difference system $\boldsymbol{H}_A$ and they are linearly independent convergent solutions of $\boldsymbol{H}_A$ when λ runs over admissible exponents associated to the initial system induced by the weight vector $\tilde{w}(\varepsilon)$.*

Proof. Since $A\lambda = s$, it is easy to show that they are formal solutions of the differential-difference system $\boldsymbol{H}_A$. We will prove that we can construct m linearly independent solutions. We note that the weight vector $\tilde{w}(\varepsilon) = (1, \dots, 1, 0) + \varepsilon v \in \mathbf{R}^{n+1}$ is in the neighborhood of $(1, \dots, 1, 0) \in \mathbf{R}^{n+1}$ and in the interior of a maximal dimensional Gröbner cone of $I_{\tilde{A}}$.

It follows from [11, p.119] that the minimal generating set of $\mathrm{in}_{(1,\dots,1,0)} I_{\tilde{A}}$ does not contain ∂_{n+1}. Since

$$\mathrm{in}_{\tilde{w}(\varepsilon)} I_{\tilde{A}} = \mathrm{in}_v(\mathrm{in}_{(1,\dots,1,0)} I_{\tilde{A}})$$

does not contain ∂_{n+1}, we have

$$M = \langle \mathrm{in}_{\tilde{w}(\varepsilon)} I_{\tilde{A}} \rangle = \langle \mathrm{in}_{w(\varepsilon)} I_A \rangle \qquad \text{in } \mathbf{C}[\partial_1, \dots, \partial_{n+1}].$$

Here, we define $w(\varepsilon)$ with $\tilde{w}(\varepsilon) = (w(\varepsilon), 0)$. Put $\tilde{\theta} = (\theta_1, \dots, \theta_{n+1})$. From [11, Theorem 3.1.3], for generic $\tilde{\beta} = (\beta_0, \beta)$, $\beta \in \mathbf{C}^d$, the initial ideal $\mathrm{in}_{(-\tilde{w}(\varepsilon), \tilde{w}(\varepsilon))} H_{\tilde{A}}(\tilde{\beta})$ is generated by $\mathrm{in}_{\tilde{w}(\varepsilon)}(I_{\tilde{A}})$ and $\tilde{A}\tilde{\theta} - \tilde{\beta}$. Let us denote by $T(M)$ the standard pairs of M. From [11, Theorem 3.2.10], the initial ideal

$$\langle \mathrm{in}_{\tilde{w}(\varepsilon)} I_{\tilde{A}}, \tilde{A}\tilde{\theta} - \tilde{\beta} \rangle \tag{3.5}$$

has $\#T(M) = \mathrm{vol}(\tilde{A})$ linearly independent solutions of the form

$$\{\tilde{x}^{\tilde{\lambda}} \mid (\partial^a, T) \in T(M)\}$$

Here, $\tilde{\lambda}$ is defined by $\tilde{\lambda}_i = a_i \in \mathbf{Z}_{\geq 0}$, $\forall i \notin T$ and $\tilde{A}\tilde{\lambda} = \tilde{\beta}$. Note that $\tilde{\lambda}$ is admissible for the d-simplex T.

Since we have

$$\langle \mathrm{in}_{\tilde{w}(\varepsilon)} I_{\tilde{A}} \rangle = \langle \mathrm{in}_{w(\varepsilon)} I_A \rangle$$

the difference between

$$\langle \mathrm{in}_{w(\varepsilon)} I_A, A\theta - \beta \rangle \tag{3.6}$$

and (3.5) is only

$$\theta_1 + \cdots + \theta_n + \theta_{n+1} - \beta_0$$

and other equations do not contain x_{n+1}, ∂_{n+1}.

For any $(\partial^a, T) \in T(M)$, we have $n+1 \in T$. Therefore, the two solution spaces (3.6) and (3.5) are isomorphic under the correspondence

$$x^\lambda \mapsto \tilde{x}^{\tilde{\lambda}} \tag{3.7}$$

Here, we put $\tilde{\lambda} = (\lambda, \lambda_{n+1})$ and λ_{n+1} is defined by

$$\sum_{i=1}^{n} \lambda_i + \lambda_{n+1} - \beta_0 = 0$$

It follows from [11, Theorem 2.3.11 and Theorem 3.2.10] that

$$\{\tilde{x}^{\tilde{\lambda}} \mid (\partial^a, T) \in T(M)\}$$

are **C**-linearly independent. Therefore, from the correspondence (3.7), the functions

$$\{x^\lambda \mid (\partial^a, T) \in T(M)\},$$

of which cardinality is $\mathrm{vol}(A)$, are **C**-linearly independent. Hence, series solutions with the initial terms

$$\left\{ \frac{x^\lambda}{\Gamma(\lambda+1)} \mid (\partial^a, T) \in T(M) \right\}$$

are **C** linearly independent, which implies the linear independence of series solutions with these starting terms [11]. We have completed the proof of the theorem and also that of Theorem 1. □

Theorem 3. *The holonomic rank of $\boldsymbol{H}_A$ is equal to the normalized volume of A.*

Proof. First we will prove $\mathrm{rank}(\boldsymbol{H}_A) \leq \mathrm{vol}(A)$. It follows from the Adolphson's theorem ([1]) that the holonomic rank of $\mathcal{A}$-hypergeometric system $H_A(\beta)$ is equal to the normalized volume of A for generic parameters β. It implies that the standard monomials for a Gröbner basis of the $\mathcal{A}$-hypergeometric system $H_A(s)$ in $\mathbf{C}(s,x)\langle \partial_1, \ldots, \partial_n \rangle$ consists of $\mathrm{vol}(A)$ elements. We note that elements in the Gröbner basis can be regarded as an element in the ring of differential-difference operators with rational function coefficients $\mathbf{U}$. We denote by ∂_j and r_j the creation and annihilation operators. The existence of them are proved in [10, Chapter 4]. Then, we have

$$H_j = \partial_j - \prod_{i=1}^{n} S_i^{-a_{ij}} \in \boldsymbol{H}_A$$

and

$$B_j = r_j - \prod_{i=1}^{n} S_i^{a_{ij}} \in \boldsymbol{H}_A, \quad r_j \in \mathbf{C}(s,x)\langle \partial_1, \ldots, \partial_n \rangle.$$

Since the column vectors of A generate the lattice $\mathbf{Z}^d$, we obtain from B_j's and H_j's elements of the form $S_i - p(s,x,\partial)$, $S_i^{-1} - q(s,x,\partial) \in \boldsymbol{H}_A$. It implies the number of standard monomials of a Gröbner basis of $\boldsymbol{H}_A$ with respect to a block order such that $S_1, \ldots, S_n > S_1^{-1}, \ldots, S_n^{-1} > \partial_1, \ldots, \partial_n$ is less than or equal to $\mathrm{vol}(A)$.

Second, we will prove $\mathrm{rank}(\boldsymbol{H}_A) \geq \mathrm{vol}(A)$. We suppose that $\mathrm{rank}(\boldsymbol{H}_A) < \mathrm{vol}(A)$ and will induce a contradiction. For the block order $S_1, \cdots, S_d > S_1^{-1}, \cdots, S_d^{-1} > \partial_1, \cdots, \partial_n$, we can show that the standard monomials T of a Gröbner basis of $\boldsymbol{H}_A$ in $\mathbf{U}$ contains only differential terms and $\#T < \mathrm{vol}(A)$ by the assumption. Let T' be the standard monomials of Gröbner basis $G(s)$ of $H_A(s)$ in the ring of differential operators with rational function coefficients $D(s)$. Note that $\#T' = \mathrm{vol}(A)$. Then T is a proper subset of the set T'. For $r \in T' \setminus T$, it follows that

$$\partial^r \equiv \sum_{\alpha \in T} c_\alpha(x,s) \partial^\alpha \qquad \mod \boldsymbol{H}_A.$$

From Theorem 2, we have convergent series solutions $f_1(s,x), \cdots, f_m(s,x)$ of $\boldsymbol{H}_A$, where $m = \mathrm{vol}(A)$. So,

$$\partial^r \bullet f_i = \sum_{\alpha \in T} c_\alpha(x,s) \partial^\alpha \bullet f_i \tag{3.8}$$

Since $f_1(s,x), \ldots, f_m(s,x)$ are linearly independent, the Wronskian standing for T'

$$W(T';f)(x,s) = \begin{vmatrix} f_1(s;x) & \cdots & f_m(\beta;x) \\ \partial^\delta f_1(s;x) & \cdots & \partial^\delta f_m(\beta;x) \\ \vdots & \cdots & \vdots \end{vmatrix} \qquad (\partial^\delta \in T' \setminus \{1\})$$

is non-zero for generic number s. However $r \in T'$ and (3.8) induce the Wronskian $W(T';f)(s,x)$ is equal to zero.

Finally, by $\mathrm{rank}(\boldsymbol{H}_A) \leq \mathrm{vol}(A)$ and $\mathrm{rank}(\boldsymbol{H}_A) \geq \mathrm{vol}(A)$, the theorem is proved. $\square$

Example 1. Put $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $\tilde{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$. This is *Airy type integral* [11, p.223].

The matrix $\tilde{A}$ is homogeneous. For $\tilde{w}(\varepsilon) = (1,1,1,0) + \frac{1}{100}(1,0,0,0)$, the initial ideal $\mathrm{in}_{\tilde{w}(\varepsilon)}(I_{\tilde{A}})$ is generated by $\partial_1^2, \partial_1\partial_2, \partial_1\partial_3, \partial_2^3$. Note that the initial ideal does not contain ∂_4. We solve the initial system $\left(\tilde{A}\tilde{\theta} - \tilde{s}\right) \bullet g = 0, \left(\mathrm{in}_{\tilde{w}(\varepsilon)}(I_{\tilde{A}})\right) \bullet g = 0$. The standard pairs (∂^a, T) for $\mathrm{in}_{\tilde{w}(\varepsilon)}(I_{\tilde{A}})$ are $(\partial_1^0\partial_2^1, \{3,4\})$, $(\partial_1^0\partial_2^0, \{3,4\})$, $(\partial_1^0\partial_2^2, \{3,4\})$. Hence, the solutions for the initial system are
$x_1^0 x_2^1 x_3^{(s_1-2)/3} x_4^{s_0-1-(s_1-2)/3}$, $x_1^0 x_2^0 x_3^{s_1/3} x_4^{a_0-s_1/3}$, $x_1^0 x_2^2 x_3^{(s_1-4)/3} x_4^{s_0-2-(s_1-4)/3}$ ([11]). Therefore, the $\mathcal{A}$-hypergeometric differential-difference system $\boldsymbol{H}_{\tilde{A}}$

has the following series solutions.

$$\tilde{\phi}_1(\tilde{\lambda},\tilde{x}) = x_4^{s_0}\left(\frac{x_2}{x_4}\right)\left(\frac{x_3}{x_4}\right)^{\frac{s_1-2}{3}} \cdot \sum_{\substack{k_1\geq 0,\ k_2\geq -1\\ (k_1,k_2)\in L'}} \frac{\left(x_1x_3^{-1/3}x_4^{-2/3}\right)^{k_1}\left(x_2x_3^{-2/3}x_4^{-1/3}\right)^{k_2}}{k_1!(k_2+1)!\Gamma(\frac{s_1-k_1-2k_2+1}{3})\Gamma(\frac{3s_0-s_1-2k_1-k_2+2}{3})}$$

$$\tilde{\phi}_2(\tilde{\lambda},\tilde{x}) = x_4^{s_0}\left(\frac{x_3}{x_4}\right)^{\frac{s_1}{3}} \cdot \sum_{\substack{k_1\geq 0,\ k_2\geq 0\\ (k_1,k_2)\in L'}} \frac{\left(x_1x_3^{-1/3}x_4^{-2/3}\right)^{k_1}\left(x_2x_3^{-2/3}x_4^{-1/3}\right)^{k_2}}{k_1!k_2!\Gamma(\frac{s_1-k_1-2k_2+3}{3})\Gamma(\frac{3s_0-s_1-2k_1-k_2+3}{3})}$$

$$\tilde{\phi}_3(\tilde{\lambda},\tilde{x}) = x_4^{s_0}\left(\frac{x_2}{x_4}\right)^2\left(\frac{x_3}{x_4}\right)^{\frac{s_1-4}{3}} \cdot \sum_{\substack{k_1\geq 0,\ k_2\geq -2\\ (k_1,k_2)\in L'}} \frac{\left(x_1x_3^{-1/3}x_4^{-2/3}\right)^{k_1}\left(x_2x_3^{-2/3}x_4^{-1/3}\right)^{k_2}}{k_1!(k_2+2)!\Gamma(\frac{s_1-k_1-2k_2-1}{3})\Gamma(\frac{3s_0-s_1-2k_1-k_2+1}{3})}$$

Here,

$L' = \{(k_1,k_2)\,|\,k_1\equiv 0 \bmod 3, k_2\equiv 0 \bmod 3\}\cup\{(k_1,k_2)\,|\,k_1\equiv 1 \bmod 3, k_2\equiv 1 \bmod 3\}$.

The matrix A is not homogeneous and by dehomogenizing the series solution for $\tilde{A}$ we obtain the following series solutions for the $\mathcal{A}$-hypergeometric differential-difference system $\boldsymbol{H}_A$.

$$\phi_1(\lambda,x) = x_2x_3^{\frac{s_1-2}{3}} \sum_{\substack{k_1\geq 0,\ k_2\geq -1\\ (k_1,k_2)\in L'}} \frac{\left(x_1x_3^{-1/3}\right)^{k_1}\left(x_2x_3^{-2/3}\right)^{k_2}}{k_1!(k_2+1)!\Gamma(\frac{s_1-k_1-2k_2+1}{3})}$$

$$\phi_2(\lambda,x) = x_3^{\frac{s_1}{3}} \sum_{\substack{k_1\geq 0,\ k_2\geq 0\\ (k_1,k_2)\in L'}} \frac{\left(x_1x_3^{-1/3}\right)^{k_1}\left(x_2x_3^{-2/3}\right)^{k_2}}{k_1!k_2!\Gamma(\frac{s_1-k_1-2k_2+3}{3})}$$

$$\phi_3(\lambda,x) = x_2^2x_3^{\frac{s_1-4}{3}} \sum_{\substack{k_1\geq 0,\ k_2\geq -2\\ (k_1,k_2)\in L'}} \frac{\left(x_1x_3^{-1/3}\right)^{k_1}\left(x_2x_3^{-2/3}\right)^{k_2}}{k_1!(k_2+2)!\Gamma(\frac{s_1-k_1-2k_2-1}{3})}$$

Here $\phi_k(x)$ is the dehomogenization of $\tilde{\phi}_k(x)$.

Finally, let us present a difference Pfaffian system for A. It can be derived by using Gröbner bases of $\boldsymbol{H}_A$ and has the following form:

$$S_1 \begin{pmatrix} f \\ x_3\partial_3 \bullet f \\ S_1 \bullet f \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{s_1 x_1}{6x_2} & \frac{3x_1x_3-4x_2^2}{6x_2x_3} & \frac{2(s_1-1)x_2+x_1^2}{6x_2} \\ \frac{s_1}{2x_2} & -\frac{3}{2x_2} & -\frac{x_1}{2x_2} \end{pmatrix} \begin{pmatrix} f \\ x_3\partial_3 \bullet f \\ S_1 \bullet f \end{pmatrix}.$$

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