ON THE RIGIDITY OF DIFFERENTIAL SYSTEMS MODELLED ON HERMITIAN SYMMETRIC SPACES AND DISPROOFS OF A CONJECTURE CONCERNING MODULAR INTERPRETATIONS OF CONFIGURATION SPACES

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Dedicated to Professor Masatake Kuranishi on his 70th birthday

Let $E(k,n;\alpha)$ be the hypergeometric system of differential equations of type (k,n) defined on the configuration space $X(k,n)$ of n hyperplanes in general position of the projective space $\mathbb{P}^{k-1}$, where α is a system of parameters:

$$\alpha=(\alpha_1,\dots,\alpha_n),\quad \alpha_1+\cdots+\alpha_n=n-k.$$

The space $X(k,n)$ is an affine set of dimension

$$m=(n-k-1)(k-1),$$

and the rank (the dimension of the linear space of solutions at a generic point) of the system $E(k,n;\alpha)$ is

$$r=\binom{n-2}{k-1}.$$

A projective solution $\varphi : X(k,n)\longrightarrow \mathbb{P}^{r-1}$ is defined by $x\mapsto u_1(x):\cdots:u_r(x)$, where the u_j's are linearly independent solutions of the system. Note that φ is multi-valued.

When $k=2$, we have

$$r=m+1;$$

so the dimension of the source space and that of the target space of the map φ agree.

When $(k,n)=(3,6)$, we have

$$r=m+2\ (=6);$$

so the image of φ is a hypersurface of $\mathbb{P}^5$.

These exhaust all the cases when the codimension of the image $Im(\varphi)$ of the projective solution φ does not exceed 1.

Consider the following integral

$$u_\Delta(x)=\int_\Delta\prod_{j=1}^{n-1} l_j(x,t)^{\alpha_j-1}dt_1\wedge\cdots\wedge dt_{k-1},$$

where $l_j(x,t)$ are defining equations of the n hyperplanes (l_n is the hyperplane at infinity) of $\mathbb{P}^{k-1}$ representing $x\in X(k,n)$, and Δ is a real $(k-1)$-dimensional twisted cycle. If $\alpha_j\notin\mathbb{Z}$, there are r cycles Δ_ν such that the u_{Δ_ν}'s are linearly independent solutions.

Notice that when $n = 2k$, the most symmetric system of parameters is given by

$$\left\{\frac{1}{2}\right\} = \left(\frac{1}{2}, \dots, \frac{1}{2}\right).$$

When $(k, n; \alpha) = (2, 4; \{1/2\})$, the following facts are classical: The integrals above are elliptic integrals, i.e., periods of elliptic curves, the equation describes the family of elliptic curves (double covers of $\mathbb{P}^1 - \{4 \text{ points}\}$), the image $Im(\varphi)$ of the projective solution φ is the upper half plane $H \subset \mathbb{P}^1$, and the map φ has a single-valued inverse so that we have the isomorphism

$$X(2, 4) \cong H/\Gamma(2),$$

where $\Gamma(2) \subset SL(2, \mathbb{Z})$ is the principal congruence subgroup of level 2.

When $(k, n; \alpha) = (3, 6; \{1/2\})$, the following is known ([MSY1]): The integrals above give periods of K3 surfaces (double covers of $\mathbb{P}^2 - \{6 \text{ lines}\}$), the equation describes a 4-dimensional family of such K3 surfaces, the image $Im(\varphi)$ of the projective solution φ lies in a non-singular quadratic hypersurface Q of $\mathbb{P}^5$, indeed it is an open dense subset of the non-compact dual $D \subset Q$ of Q, and that φ has a single-valued inverse map so that we have the isomorphism

$$X(3, 6) \cong (D - \{\text{fixed points of } \Gamma\})/\Gamma,$$

where Γ is an arithmetic subgroup of the group of automorphisms of D.

Since Q can be regarded as the Grassmannian variety $Gr_{2,4}$, and since the Grassmannian $Gr_{k-1,n-2}$ can be equivariantly and minimally embedded in $\mathbb{P}^{r-1}$, we are very happy if $Im(\varphi)$ might lie in $Gr_{k-1,n-2} \subset \mathbb{P}^{r-1}$.

Especially when $(k, n; \alpha) = (4, 8; \{1/2\})$, many mathematicians are expecting that $Im(\varphi)$ would lie in $Gr_{3,6} \subset \mathbb{P}^{20-1}$, and that we get a nice isomorphism like the examples above. Because the system describes a 9-dimensional family of Calabi-Yau 3-folds (double covers of $\mathbb{P}^3 - \{8 \text{ planes}\}$), it is a hot topic now. Notice that the integral above gives periods of such 3-folds.

We are very sorry to declare the following

Theorem 1. *If $k \geqq 3$, $n - k \geqq 3$ and $(k, n) \neq (3, 6)$, then the image $Im(\varphi)$ of the projective solution of the system $E(k, n; \alpha)$ does not lie in $Gr_{k-1,n-2} \subset \mathbb{P}^{r-1}$ for any α_j.*

The proof is given by showing that the system $E(k, n)$ is not equivalent to the system of differential equations defining the Plücker embedding of $Gr_{k,1,n-2}$. The actual key to prove inequivalence is the computation of certain Lie algebra cohomology, which due to Se-ashi reduces the problem to the comparison of the symbols of both systems.

In Sections 1 and 2 we review the equivalence problem of differential systems and prove a general result on rigidity of differential systems modelled on equivariant projective embedding of the hermitian symmetric spaces (Corollary 3). The comparison of the symbols will be given in Seection 3. In Section 4 we provide a much simpler proof of inequivalence valid for $E(4, 8)$.

Acknowledgment: When the first and the third authors were preparing the paper [MSY1], they dreamed about the story of $E(4, 8; \{1/2\})$ analogous to $E(3, 6; \{1/2\})$. It was disproved soon; they were disappointed and had no idea to publish this negative fact. After Professor Y. Se-ashi's unexpected death, his notes were completed by the second author, who pointed out that the conjecture could be disproved generally by following the line of the completed note. Meanwhile several mathematicians asked the third author whether the image of the

projective solution of $E(4,8;\{1/2\})$ is in $Gr_{3,6}$, moreover some of them showed him (sketchy) proofs. So we decided to publish this negative result.

1. Projective embedding of hermitian symmetric spaces

As we explained in [MSY2], it is classically well known that a system R in m variables of rank r is nothing but an m-dimensional submanifold M in $\mathbb{P}^{r-1}$; more precisely, two such systems are said to be equivalent if one is transformed into the other by a change of independent variables and by the replacement of the unknown by its product with a non-zero function and we have the bijective correspondence

$$\begin{aligned}&\{\text{germs of systems in } m \text{ variables of rank } r\}/\text{equivalence}\\ \leftrightarrow\ &\{\text{germs of } m\text{-dimensional submanifolds in } \mathbb{P}^{r-1}\}\ /\ PGL(r)\end{aligned}$$

by associating to a system R the image M of its projective solution.

As for the system $E(3,6;\{1/2\})$, we checked in [MSY1] that the image of the projective solution lies in a non-singular quadratic hypersurface Q by utilizing the projective hypersurface theory in $\mathbb{P}^5$.

Our concern in this paper is the Grassmannian variety $Gr_{k-1,n-2}$ in $\mathbb{P}^{r-1}$ embedded as the image of the Plücker embedding, on the lower side of the above correspondence. Hence, in this section, we would like to construct group-theoretically a system $R(k,n)$ in m variables of rank r, which corresponds to $Gr_{k-1,n-2}$ in $\mathbb{P}^{r-1}$ in the above diagram, where $m=(n-k-1)(k-1)$ and $r=\binom{n-2}{k-1}$, and we discuss the inequivalence of $E(k,n)$ and $R(k,n)$ in §3 by virtue of Se-ashi's theory for the equivalence of integrable linear differential equations of finite type.

For this purpose and also as a motivation to introduce Se-ashi's theory in §2, which in fact enables us to construct $R(k,n)$ a little generally, we will consider here projective embedding of hermitian symmetric spaces.

Group-theoretically, a compact irreducible hermitian symmetric space M corresponds to a simple graded Lie algebra of the first kind as follows: Let $\mathfrak{l}=\mathfrak{l}_{-1}\oplus\mathfrak{l}_0\oplus\mathfrak{l}_1$ be a simple graded Lie algebra of the first kind, i.e.,

(i) $\mathfrak{l}$ is a simple Lie algebra over $\mathbb{C}$.

(ii) $\mathfrak{l}=\mathfrak{l}_{-1}\oplus\mathfrak{l}_0\oplus\mathfrak{l}_1$ is a vector space direct sum such that $\mathfrak{l}_{-1}\neq\{0\}$.

(iii) $[\mathfrak{l}_p,\mathfrak{l}_q]\subset\mathfrak{l}_{p+q}$, where $\mathfrak{l}_p=\{0\}$ for $|p|\geqq 2$.

Let L be the simply connected Lie group with Lie algebra $\mathfrak{l}$ and L' be the analytic subgroup of L with Lie algebra $\mathfrak{l}'=\mathfrak{l}_0\oplus\mathfrak{l}_1$. Then $M=L/L'$ is a compact (irreducible) hermitian symmetric space and every compact irreducible hermitian symmetric space is obtained in this manner from a simple graded Lie algebra of the first kind. M is called the *model space* associated with $\mathfrak{l}=\mathfrak{l}_{-1}\oplus\mathfrak{l}_0\oplus\mathfrak{l}_1$. For example, when $M=Gr_{k-1,n-2}$, we have $\mathfrak{l}=\mathfrak{sl}(n-2,\mathbb{C})$ and the gradation $\mathfrak{l}=\mathfrak{l}_{-1}\oplus\mathfrak{l}_0\oplus\mathfrak{l}_1$ is given by subdividing matrices as follows:

$$(1.1)\qquad \mathfrak{l}_{-1}=\left\{\begin{pmatrix}0&0\\C&0\end{pmatrix}\mid C\in M(p,i)\right\},\ \mathfrak{l}_1=\left\{\begin{pmatrix}0&D\\0&0\end{pmatrix}\mid D\in M(i,p)\right\},$$

$$\mathfrak{l}_0=\left\{\begin{pmatrix}A&0\\0&B\end{pmatrix}\mid A\in M(i,i),\ B\in M(p,p)\text{ and }\mathrm{tr}A+\mathrm{tr}B=0\right\}.$$

where $i = k-1$, $p = n-k-1$ and $M(a,b)$ denotes the set of $a \times b$ matrices.

An equivariant projective embedding of the model space $M = L/L'$ can be obtained from an irreducible representation of L as follows: Let $\tau : L \to GL(T)$ be an irreducible representation of L with the highest weight Λ. Let t_Λ be a maximal vector in T of the highest weight Λ. Then a stabilizer of the line $[t_\Lambda]$ spanned by v_Λ in T is a parabolic subgroup of L. When this stabilizer coincides with L', we obtain an equivariant projective embedding of $M = L/L'$ by taking the L-orbit passing through $[t_\Lambda]$ in the projective space $P(T)$ consisting of all lines in T passing through the origin. For example, when $M = Gr_{k-1,n-2}$, we take the exterior representation τ_0 of $L = SL(n-2, \mathbb{C})$ on $T = \bigwedge^{k-1} \mathbb{C}^{n-2}$:

$$\tau_0 : SL(n-2, \mathbb{C}) \to GL(\wedge^{k-1}\mathbb{C}^{n-2}),$$

where $\tau_0(a)(v_1 \wedge \cdots \wedge v_{k-1}) = a(v_1) \wedge \cdots \wedge a(v_{k-1})$ for $a \in SL(n-2, \mathbb{C})$ and $v_i \in \mathbb{C}^{n-2}$ $(i = 1, 2, \ldots, n-1)$. Let $\{e_1, \ldots, e_{n-2}\}$ be the natural basis of $\mathbb{C}^{n-2}$. Then τ_0 is an irreducible representation of $SL(n-2, \mathbb{C})$ with the maximal vector $e_1 \wedge \cdots \wedge e_{k-1}$ for a suitable choice of a Cartan subalgebra and a simple root system of $\mathfrak{sl}(n-2, \mathbb{C})$. From (1.1), we see that the stabilizer of the line $[e_1 \wedge \cdots \wedge e_{k-1}]$ coincides with L'. Thus we see that the Plücker embedding of $Gr_{k-1,n-2}$ is obtained from the irreducible representation τ_0 of $SL(n-2, \mathbb{C})$.

Next, for an irreducible representation $\tau : L \to GL(T)$, we will construct a (positive) line bundle F over M such that the above orbit is obtained as an embedding of M by global sections of F. To construct F, let us take the dual representation $\rho : L \to GL(S)$ of τ, i.e., $S = T^*$ is the dual space of T and $\rho = \tau^*$ is defined by

$$\langle \rho(g)(\xi), t \rangle = \langle \xi, \tau(g^{-1})(t) \rangle,$$

for $g \in L, t \in T, \xi \in T^*$ and $\langle \ , \ \rangle$ is the canonical pairing between T^* and T. Then, when τ is an irreducible representation with the highest weight Λ (for a fixed choice of a Cartan subalgebra and a simple root system of $\mathfrak{l}$), ρ is the irreducible representation with the lowest weight $-\Lambda$. Let us take a basis $\{ t_1, \ldots, t_r \}$ of T consisting of weight vectors of τ such that $t_1 = t_\Lambda$. Then the dual basis $\{ s_1, \ldots, s_r \}$ of $\{ t_1, \ldots, t_r \}$ in $S = T^*$ consists of weight vectors of ρ and s_1 is a weight vector corresponding to $-\Lambda$. Let W and W' be the subspaces of S spanned by a vector s_1 and by vectors $s_2, \ldots, s_r$, respectively. Since L' is the stabilizer of the line $[t_1]$, W' is preserved by L'. Hence we get the representation ρ_W of L':

$$\rho_W : L' \to GL(W),$$

through the projection $\pi_0 : S = W \oplus W' \to W$.

Relative to the representation ρ_W, L' acts on $L \times W$ on the right by

$$(g, w)g' = (gg', \rho_W(g')^{-1}(w)),$$

for $g \in L, w \in W$ and $g' \in L'$. Then $F = L \times W/L'$ is the line bundle over $M = L/L'$.

As is well known, the space $\Gamma(F)$ of global sections of F is identified with the space $\mathcal{F}(L, W)_{L'}$ of all W-valued functions f on L satisfying

$$f(gg') = \rho_W(g')^{-1} f(g),$$

for $g \in L$ and $g' \in L'$, via the correspondence $f \in \mathcal{F}(L, W)_{L'} \mapsto \sigma_f \in \Gamma(F)$ given by

$$\sigma_f(\pi_1(g)) = \pi_2(g, f(g)),$$

where $\pi_1 : L \to M = L/L'$ and $\pi_2 : L \times W \to F$ denote the natural projections. Then each $s \in S$ defines an element $\sigma_s \in \Gamma(F)$ via the above correspondence by

$$f_s(g) = \pi_0(\rho(g^{-1})s)$$

for $g \in L$.

Now let us check that global sections of F give the desired embedding of M into $P(T)$. We utilize the above basis $\{ t_1, \dots, t_r \}$ and $\{ s_1, \dots, s_r \}$ of T and $S = T^*$. Let us consider a map $\hat{\varphi}$ of L into T defined by

$$\hat{\varphi}(g) = \sum_{i=1}^{r} \langle f_{s_i}(g), t_1 \rangle t_i \tag{1.2}$$

for $g \in L$. Then, from $\langle f_{s_i}(g), t_1 \rangle = \langle \rho(g^{-1}) s_i, t_1 \rangle$, $\hat{\varphi}$ induces a map φ of M into $P(T)$ satisfying the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\hat{\varphi}} & T \backslash \{0\} \\ \downarrow & & \downarrow \\ M = L/L' & \xrightarrow{\varphi} & P(T). \end{array}$$

For $g \in L$, if we represent $\tau(g)$ as a matrix A with respect to the basis $\{ t_1, \dots, t_r \}$, $\rho(g^{-1})$ is represented by the transposed matrix ${}^tA$ of A with respect to the basis $\{ s_1, \dots, s_r \}$.From (1.2), $\hat{\varphi}(g)$ corresponds to the first row vector of ${}^tA$. Hence we obtain

$$\hat{\varphi}(g) = \tau(g)(t_1).$$

Thus the image of φ coincides with the L-orbit passing through $[t_1]$ in $P(T)$.

Owing to Se-ashi's theory, which will be discussed in the next section, we can construct a system R_ρ of linear differential equations of rank r on F such that every local solution of R_ρ is a restriction of σ_s for some $s \in S$ as in the following: Let $J^p(F)$ be the bundle of p-jets of F. The fiber $J^p_x(F)$ of $J^p(F)$ over a point x of M is the quotient of the space of germs of sections of F at x by the subspace of germs which vanish to order $p+1$ at x. Let $\pi^p_q : J^p(F) \to J^q(F)$ denote the natural projection for $p > q$. At each point $x \in M = L/L'$, let $(R^\rho_p)_x$ be the subspace of $J^p_x(F)$ defined by

$$(R^\rho_p)_x = \{ j^p_x(\sigma_s) \mid s \in S \},$$

where $j^p_x(\sigma_s)$ is the p-jet at x of the section σ_s. Let R^ρ_p be the subbundle of $J^p(F)$ defined by

$$R^\rho_p = \bigcup_{x \in M} (R^\rho_p)_x.$$

Then there exists a natural number p_0 such that π^p_{p-1} induces a bundle isomorphism of R^ρ_p onto R^ρ_{p-1} for every $p \geqq p_0$ (for more detail, see §2.2). Putting $R^\rho = R^\rho_{p_0}$, we see that R^ρ has the desired property. In fact, R^ρ is the model equation for the typical symbol of type $(\mathfrak{l}, \rho)$ in Se-ashi's theory (see Proposition in §2.3).

We denote by $R(k, n)$ the system constructed as above from the exterior representation ρ_0 of $L = SL(n-2, \mathbb{C})$ on $S = \bigwedge^{n-k-1} \mathbb{C}^{n-2}$, which is dual to the representation τ_0. Then, from the construction, the projective solution of $R(k,n)$ coincides with the Plücker embedding of $M = Gr_{k-1,n-2}$. Thus we obtain the system in m variables of rank r corresponding to $Gr_{k-1,n-2}$ in $\mathbb{P}^{r-1}$ in the bijective correspondence given at the beginning of this section. We shall examine the symbol of $R(k,n)$ in detail and discuss the inequivalence of $E(k,n)$ and $R(k,n)$ in §3.

2. Se-ashi's Theorem

Se-ashi's theory on the equivalence of integrable linear differential equations of finite type deals with the special classes of equations characterized by their symbols, namely, with those equations having the typical symbol of type $(\mathfrak{l}, \rho)$, where ρ is an irreducible representation of a (semi-)simple graded Lie algebra $\mathfrak{l}$ of the first kind. We will briefly review his theory and also prove a theorem on the Lie algebra cohomology, which was left unpublished in his note. We will confine ourselves in the holomorphic category and take $\mathfrak{l}$ to be a simple Lie algebra over $\mathbb{C}$ in the following argument, although his theory applies also in the real C^∞ category and for semi-simple Lie algebras over $\mathbb{R}$.

2.1. Linear differential equations of finite type. Let us begin with recalling some generalities on jet bundles. Let M be a manifold of dimension m. We denote by T and T^* the tangent and the cotangent bundle of M respectively. For a vector bundle E over M, we denote by $J^p(E)$ the bundle of p-jets of E. The fibre of $J^p(E)$ over a point x of M is the quotient of the space of germs of sections of E at x by the subspace of germs which vanish to order $p+1$ at x. We identify $J^0(E)$ with E and put $J^{-1}(E) = M$ for convention. Let π^p_q denote the natural projection of $J^p(E)$ onto $J^q(E)$ for $p > q$. For a section s of E, its p-th jet at x is denoted by $j^p_x(s)$. There exist the natural vector bundle morphism $\varepsilon_p : S^pT^* \otimes E \to J^p(E)$ and the exact sequence

$$0 \longrightarrow S^pT^* \otimes E \xrightarrow{\varepsilon_p} J^p(E) \xrightarrow{\pi^p_{p-1}} J^{p-1}(E) \longrightarrow 0,$$

where S^pT^* denotes the p-th symmetric product of T^*.

A subbundle R of $J^p(E)$ is called a system of (homogeneous) linear differential equations of order p on E. A solution of R is a (local) section s of E satisfying $j^p_x(s) \in R_x$ at each $x \in M$. Let $R_r = \pi^p_r(R)$ be the image of the projection of R into $J^r(E)$ and put $\mathfrak{g}_r = R_r \cap (S^rT^* \otimes E)$ for $r \leqq p$, which is called the r-th *symbol* of R. We have an exact sequence

$$0 \longrightarrow \mathfrak{g}_r \xrightarrow{\varepsilon_r} R_r \xrightarrow{\pi^r_{r-1}} R_{r-1} \longrightarrow 0.$$

The direct sum $S_x = \bigoplus_{r=0}^p (\mathfrak{g}_r)_x$ is called the (total) symbol of R at $x \in M$, where $(\mathfrak{g}_r)_x \subset S^rT^*_x \otimes E_x$ denotes the fibre of $\mathfrak{g}_r$ over x.

A system R of order p is said to be of finite type if $\mathfrak{g}_p = 0$, i.e., if $\pi^p_{p-1} : R \to R_{p-1}$ is an isomorphism. A system R of finite type is said to be integrable if, for each $\eta \in R$, there is a (local) solution s for which $j^p_x(s) = \eta$, where $x = \pi^p_{-1}(\eta)$. In this case, such a solution s is uniquely determined by the initial condition $\eta \in R_x$. Thus, by a continuation of solutions along a curve $x_t, t \in [0,1]$ on M, we get a parallel displacement $\tau : R_{x_0} \to R_{x_1}$. Namely, for each $\eta_0 \in R_{x_0}$, we take a local solution s of R such that $j^p_{x_0}(s) = \eta_0$, continue this solution along x_t and put $\tau(\eta_0) = \eta_1 = j^p_{x_1}(s) \in R_{x_1}$. In this manner, we obtain a connection ∇ in the vector bundle R over M. Since the above parallel displacement is independent of curves joining x_0 and x_1 in a neighborhood of x_0, ∇ is a flat connection. In fact, ∇ is induced from the Spencer operator acting on $J^p(E)$ (Proposition 1.5.1 [S]).

Let E and E' be vector bundles over M. Let R and R' be systems of order p on E and E', respectively. Then a bundle isomorphism $\phi : E \to E'$ is called an isomorphism of R onto R' if $J^p(\phi)$ maps R onto R', where $J^p(\phi) : J^p(E) \to J^p(E')$ is the lift of ϕ. In this case we denote by $R^p(\phi)$ the restriction of $J^p(\phi)$ to R. Obviously, $R^p(\phi)$ is a vector bundle isomorphism of R onto R', which preserves the flat connections in R and R'.

2.2. Typical symbol of type $(\mathfrak{l}, \rho)$**.** Let R be a system of linear differential equations of order p on E and let $\mathfrak{g}_r$ be the r-th symbol of R for $r = 0, \dots, p$. We fix vector spaces V and W over $\mathbb{C}$ such that $\dim V = \dim M$ and $\dim W = \mathrm{rank} E$, respectively. Let $S = \bigoplus_{r=0}^{p} S_r$ be a graded vector subspace of $\bigoplus_{r=0}^{p} S^r V^* \otimes W$. Then the system R is said to be of type S if, for each $x \in M$, there exist linear isomorphisms $z_T : V \cong T_x$ and $z_E : W \cong E_x$ such that the induced isomorphism $({}^t z_T^{-1}) \otimes z_E : S^r V^* \otimes W \cong S^r T_x^* \otimes E_x$ sends S_r onto $(\mathfrak{g}_r)_x$ for every r. In this case, S is called the *typical symbol* of R.

Now we introduce the important classes of typical symbols for integrable systems of linear differential equations of finite type in the following.

Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra over $\mathbb{C}$ of the first kind and $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$ an irreducible representation of $\mathfrak{l}$ on a vector space S.

As is well-known, there exists a unique element $Z \in \mathfrak{l}_0$ (Lemma 4.1.1. [S]) such that

$$\mathfrak{l}_p = \{\, X \in \mathfrak{l} \mid [Z, X] = pX \,\} \qquad (p = -1, 0, 1).$$

Z is called the characteristic element of $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Since $ad(Z)$ is a semi-simple endomorphism with eigenvalues -1, 0 and 1, $\rho(Z)$ is a semi-simple endomorphism of S (Corollary 6.4 [Hu]) with real eigenvalues (see the arguments in §2.5). Moreover, putting $S_{(\mu)} = \{\, s \in S \mid \rho(Z)(s) = \mu s \,\}$, we have

$$\rho(\mathfrak{l}_p) S_{(\mu)} \subset S_{(\mu + p)} \qquad \text{for} \quad p = -1, 0, 1.$$

Let λ_0 be the minimum eigenvalue of $\rho(Z)$ and put $S_r = S_{(\lambda_0 + r)}$ for $r \geqq 0$. Then, since ρ is irreducible, there exists a natural number p_0 (Proposition 4.2.1 [S]) such that $S_r \neq \{0\}$ for $r = 0, 1, \dots, p_0 - 1$ and

$$S = \bigoplus_{r=0}^{p_0 - 1} S_r.$$

For each integer q $(0 \leqq q < p_0)$ put $S_q(q) = \{\, s \in S_q \mid \rho(\mathfrak{l}_{-1})(s) = 0 \,\}$. Then $S_0(0) = S_0$ and $S_q(q)$ is a $\rho(\mathfrak{l}_0)$-invariant subspace of S_q. We define a linear subspace $S(q) = \bigoplus_{q \leqq r < p_0} S_r(q)$ of S inductively by

$$S_{r+1}(q) = \rho(\mathfrak{l}_1)(S_r(q)) \subset S_{r+1}.$$

One can easily check that $S_r(q)$ is $\rho(\mathfrak{l}_0)$-invariant and $\rho(\mathfrak{l}_{-1})(S_{r+1}(q)) \subset S_r(q)$ by induction on $r \geqq q$. Thus $S(q)$ is a $\rho(\mathfrak{l})$-submodule of S. Since ρ is irreducible, we get $S(0) = S$ and $S(q) = 0$ for $q > 0$. Hence, putting $S_r = \{0\}$ for $r \geqq p_0$, we obtain

$$S_0 = \{\, s \in S \mid \rho(\mathfrak{l}_{-1})(s) = 0 \,\}, \tag{2.1}$$

and

$$S_{r+1} = \rho(\mathfrak{l}_1)(S_r) \qquad \text{for} \quad r \geqq 0. \tag{2.2}$$

Now we put $V = \mathfrak{l}_{-1}$ and $W = S_0$. Then we have a linear isomorphism ι_r of S_r into $S^rV^* \otimes W$ $(r = 1, \ldots, p_0 - 1)$ defined by

$$\iota_r(s)(X_1, \ldots, X_r) = (-1)^r \rho(X_1) \cdots \rho(X_r)(s).$$

Since $\mathfrak{l}_{-1}$ is abelian, ι_r is well-defined. In this manner, $S = \bigoplus_{r \geqq 0} S_r$ is regarded as a graded vector subspace of $\bigoplus_{r \geqq 0} S^rV^* \otimes W$, which is called the *typical symbol of type* $(\mathfrak{l}, \rho)$.

As an example, we construct the typical symbol of type $(\mathfrak{l}, \rho)$, when $\mathfrak{l} = \mathfrak{sl}(n-2, \mathbb{C})$ is endowed with the gradation given in (1.1) and $\rho = \rho_0$ is the exterior representation on $S = \wedge^{n-k-1}\mathbb{C}^{n-2}$:

$$\rho : \mathfrak{sl}(n-2, \mathbb{C}) \to \mathfrak{gl}(\wedge^{n-k-1}\mathbb{C}^{n-2}),$$

where

$$\rho(X)(v_1 \wedge \cdots \wedge v_{n-k-1}) = \sum_{i=1}^{n-k-1} v_1 \wedge \cdots \wedge X(v_i) \wedge \cdots \wedge v_{n-k-1}$$

for $X \in \mathfrak{sl}(n-2, \mathbb{C})$ and $v_i \in \mathbb{C}^{n-2}$ $(i = 1, 2, \ldots, n-k-1)$.

Let $\{e_1, \ldots, e_{n-2}\}$ be the natural basis of $\mathbb{C}^{n-2}$. Then $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is the isotropy (stabilizer) algebra of the line $[e_1 \wedge \cdots \wedge e_{k-1}]$ in $\wedge^{k-1}\mathbb{C}^{n-2}$. We denote by $E_{ab} \in \mathfrak{gl}(n-2, \mathbb{C})$ $(1 \leqq a, b \leqq n-2)$ the matrix whose (a, b)-component is 1 and all of whose other components are 0. From (1.1), we have the following basis for $V = \mathfrak{l}_{-1}$ and $\mathfrak{l}_1$:

$$\begin{aligned} V = \mathfrak{l}_{-1} =& \langle E_{pi} \mid 1 \leqq i \leqq k-1,\ k \leqq p \leqq n-2 \rangle \\ \mathfrak{l}_1 =& \langle E_{ip} \mid 1 \leqq i \leqq k-1,\ k \leqq p \leqq n-2 \rangle \end{aligned}$$

Since $E_{pi}(e_j) = \delta_{ij} e_p$ for $1 \leqq j \leqq k-1$ and, $E_{pi}(e_q) = 0$ for $k \leqq q \leqq n-2$, we have from (2.1)

$$W = S_0 = \langle e_k \wedge \cdots \wedge e_{n-2} \rangle.$$

For $1 \leqq i_1 < \cdots < i_r \leqq k-1$ and $k \leqq p_1 < \cdots < p_r \leqq n-2$, we put

$$e(p_1, \ldots, p_r) = e_k \wedge \cdots \wedge \widehat{e}_{p_1} \wedge \cdots \wedge \widehat{e}_{p_r} \wedge \cdots \wedge e_{n-2} \in \wedge^{n-k-r-1}\mathbb{C}^{n-2},$$

and consider the following element of S:

$$s(i_1, \ldots, i_r, p_1, \ldots, p_r) = e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e(p_1, \ldots, p_r) \in S = \wedge^{n-k-1}\mathbb{C}^{n-2}.$$

Then, from (2.2) and $E_{ip}(e_j) = 0$, $E_{ip}(e_q) = \delta_{pq} e_i$ for $1 \leqq j \leqq k-1$, $k \leqq q \leqq n-2$, we get

$$S_r = \langle s(i_1, \ldots, i_r, p_1, \ldots, p_r) \mid 1 \leqq i_1 < \cdots < i_r \leqq k-1, k \leqq p_1 < \cdots < p_r \leqq n-2 \rangle,$$

for $r = 1, 2, \ldots, p_0 - 1$ and

$$S_r = \{0\},$$

for $r \geqq p_0 = \min\{k, n-k\}$. Moreover, for $X = \sum_{ip} X_{ip} E_{pi} \in V$, we have

$$\begin{aligned} \iota_r(s(i_1, \ldots, i_r, p_1, \ldots, p_r))(X, \ldots, X) &= r!(-1)^r X(e_{i_1}) \wedge \cdots \wedge X(e_{i_r}) \wedge e(p_1, \ldots, p_r) \\ &= r!(-1)^r (\sum_{\sigma} \operatorname{sgn} \sigma\, X_{i_1 p_{\sigma(1)}} \cdots X_{i_r p_{\sigma(r)}})\, e_{p_1} \wedge \cdots \wedge e_{p_r} \wedge e(p_1, \ldots, p_r). \end{aligned}$$

Thus, by fixing a basis of W and identifying SV^* with the ring of polynomials on V, we see that $S_1 = V^*$ and $S_r \subset S^r V^*$ is spanned by the minor determinants of degree r of the matrix (X_{ip}), which are the linear coordinates of V.

2.3. Model systems. Starting from the typical symbol $S = \bigoplus_{r=0}^{p} S_r \subset \bigoplus_{r=0}^{p} S^r V^* \otimes W$ with the properties $S_0 = W$ and $S_p = 0$, we now explain a recipe to construct an integrable system of differential equations of finite type of order p modeled after S.

The construction of the model system R_S is preceded by the consideration of the Lie algebra $\mathfrak{g}$ of infinitesimal automorphisms of the constant coefficient differential equations modeled after S.

Let $E_0 = V \times W$ be the trivial bundle over the vector space V. Then the fibre $J_0^p(E_0)$ of $J^p(E_0)$ at the origin $0 \in V$ is identified with $\bigoplus_{r=0}^{p} S^r V^* \otimes W$, where $S^r V^* \otimes W$ can be regarded as the set of W-valued homogeneous polynomials of degree r on V. Thus, starting from the typical symbol $S = \bigoplus_{r=0}^{p} S_r \subset \bigoplus_{r=0}^{p} S^r V^* \otimes W$, our first (local) model is the constant coefficient differential equations given as the subbundle $\hat{R}_S = V \times S$ of $J^p(E_0)$, whose solutions consist of W-valued polynomials contained in $S \subset SV^* \otimes W$.

Let us consider an infinitesimal bundle automorphism of E_0 preserving $\hat{R}_S$. An infinitesimal bundle automorphism of E_0 has a form

$$\sum_i \xi^i(x) \frac{\partial}{\partial x^i} + \sum_{\alpha,\beta} A_{\alpha,\beta}(x) y^\beta \frac{\partial}{\partial y^\alpha},$$

where (x^i) and (y^α) are linear coordinates of V and W, respectively. Thus the Lie algebra $\mathfrak{a}$ of (formal) infinitesimal bundle automorphisms of E_0 can be expressed as a graded Lie algebra $\mathfrak{a} = \bigoplus_{r \geqq -1} \mathfrak{a}_r$ by putting

$$\mathfrak{a}_r = S^{r+1} V^* \otimes V \oplus S^r V^* \otimes \mathfrak{gl}(W),$$

where $\mathfrak{a}_{-1} = V$ corresponds to constant coefficient vector fields on V. The bracket operation in $\mathfrak{a}$ is given by

$$\begin{aligned}
[f \otimes v,\, g \otimes w\,] &= -f(i(v)g) \otimes w + g(i(w)f) \otimes v, \\
[f \otimes A, g \otimes w\,] &= g(i(w)f) \otimes A, \\
[f \otimes A, g \otimes B] &= fg \otimes [A, B],
\end{aligned}$$

where $f, g \in SV^*$, $v, w \in V$ and $A, B \in \mathfrak{gl}(W)$; $i(v)$ denotes the inner multiplication. The Lie algebra $\mathfrak{a}$ acts naturally on the space $SV^* \otimes W$ that is regarded as the space of cross sections of E_0:

$$(f \otimes v + g \otimes A)(h \otimes w) = -f(i(v)h) \otimes w + gh \otimes A(w),$$

where $f, g, h \in SV^*$, $v, w \in V$ and $A \in \mathfrak{gl}(W)$.

Then the Lie algebra $\mathfrak{g}$ of infinitesimal automorphisms of $\hat{R}_S$ is given by

$$\mathfrak{g} = \{\, X \in \mathfrak{a} \mid X(S) \subset S \,\}.$$

$\mathfrak{g}$ is a graded subalgebra of $\mathfrak{a} = \bigoplus_{r \geqq -1} \mathfrak{a}_r$, i.e., $\mathfrak{g} = \bigoplus_{r \geqq -1} \mathfrak{g}_r$, where $\mathfrak{g}_r = \mathfrak{g} \cap \mathfrak{a}_r$. The Lie algebra $\mathfrak{gl}(S)$ has also the gradation given by

$$\mathfrak{gl}(S)_r = \{\, X \in \mathfrak{gl}(S) \mid X(S_l) \subset S_{l+r} \quad \text{for any } l \,\}.$$

Referring the action above we have a restriction homomorphism: $\mathfrak{g} \to \mathfrak{gl}(S)$, which sends $\mathfrak{g}_r$ into $\mathfrak{gl}(S)_r$. Assume here the following two conditions for S, which are satisfied by the typical symbol of type $(\mathfrak{l}, \rho)$:

$$\begin{aligned} &(A1) \quad \text{The action of } \mathfrak{a}_{-1} = V \text{ leave } S \text{ invariant.} \\ &(A2) \quad \text{The action of } \mathfrak{a}_{-1} = V \text{ on } S \text{ is faithful.} \end{aligned}$$

Then this homomorphism turns out to be injective and we can characterize $\mathfrak{g}_r$ as a subspace of $\mathfrak{gl}(S)_r$ as follows:

$$\begin{aligned} \mathfrak{g}_{-1} &= V, \\ \mathfrak{g}_r &= \{\, X \in \mathfrak{gl}(S)_r \mid [\mathfrak{g}_{-1}, X] \subset \mathfrak{g}_{r-1} \,\} \qquad \text{for} \quad r \geqq 0. \end{aligned} \tag{2.3}$$

Put $\mathfrak{u}_r = S^r V^* \otimes \mathfrak{gl}(W) \subset \mathfrak{a}_r$. Then $\mathfrak{u} = \bigoplus_{r \geqq 0} \mathfrak{u}_r$ is an ideal of $\mathfrak{a}$ and $\mathfrak{n} = \mathfrak{u} \cap \mathfrak{g}$ is an ideal of $\mathfrak{g}$. We can see

$$\mathfrak{n}_r = \{\, X \in \mathfrak{gl}(S)_r \mid [\mathfrak{g}_{-1}, X] \subset \mathfrak{n}_{r-1} \,\} \qquad \text{for} \quad r \geqq 0, \tag{2.4}$$

where we put $\mathfrak{n}_{-1} = \{0\}$ for convention.

In the case of the typical symbol of type $(\mathfrak{l}, \rho)$, we have the following : We identify $\mathfrak{l}$ with its image $\rho(\mathfrak{l})$ in $\mathfrak{gl}(S)$ as follows. Let $\mathfrak{c}$ denote the centralizer of $\mathfrak{l}$ in $\mathfrak{gl}(S)$ and $\mathfrak{g}^\perp$ the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{gl}(S)$ with respect to the non-degenerate bilinear form Tr given by $\mathrm{Tr}(X, Y) = \operatorname{trace} XY$ for $X, Y \in \mathfrak{gl}(S)$. Then, from (2.3) and (2.4), we have (Proposition 4.4.1 [S])

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{c}, \quad \mathfrak{n} = \mathfrak{c}, \qquad \mathfrak{gl}(S) = \mathfrak{l} \oplus \mathfrak{n} \oplus \mathfrak{g}^\perp \quad \text{(Tr-orthogonal)}. \tag{2.5}$$

In fact, since ρ is irreducible, $\mathfrak{c}$ coincides with the center of $\mathfrak{gl}(S)$ in our case.

Now let $S = \bigoplus_{r=0}^{p} S_r$ be a typical symbol satisfying $S_0 = W$, $S_p = 0$, and the above conditions $(A.1)$ and $(A.2)$. Then the model system R_S is constructed as follows: We filtrate the space S by subspaces $S^r = \bigoplus_{l=r}^{p} S_l$. Notice that the group $GL(V) \times GL(W)$ acts on $\mathfrak{a}$ by the adjoint action: for $a \in GL(V) \times GL(W)$ and $X \in \mathfrak{a}$, the action is $(aX)(s) = (a \cdot X \cdot a^{-1})(s)$ for $s \in S$. Let us define groups

$$\begin{aligned} G_0 &= \{\, a \in GL(V) \times GL(W) \mid a(S) \subset S \,\}, \\ GL^{(0)}(S) &= \{\, g \in GL(S) \mid g(S^r) \subset S^r \quad \text{for any r} \,\}. \end{aligned}$$

Let $\widetilde{G}$ be the analytic subgroup of $GL(S)$ with Lie algebra $\mathfrak{g} \in \mathfrak{gl}(S)$ and put

$$\begin{aligned} G &= \widetilde{G} \cdot G_0, \\ G' &= G \cap GL^{(0)}(S). \end{aligned}$$

We see that the groups G_0 and G' are Lie subgroups of $GL(S)$ with Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}' = \bigoplus_{r \geqq 0} \mathfrak{g}_r$ respectively. Since G' preserves the filtration $\{S^r\}_{r \geqq 0}$ of S, we get the representation ρ_W of G':

$$\rho_W : G' \to GL(W),$$

through the projection $\pi_0 : S = \bigoplus_{r=0}^{p} S_r \to S_0 = W$.

Let E_S be the vector bundle over $M = G/G'$ associated with the representation $\rho_W : G' \to GL(W)$; G' acts on $G \times W$ on the right by

$$(g, w)g' = (gg', \rho_W(g')^{-1}(w)),$$

for $g \in G, w \in W$ and $g' \in G'$. Then E_S is the vector bundle over $M = G/G'$ defined by $E_S = G \times W/G'$. As in §1, each $s \in S$ defines an element $\sigma_s \in \Gamma(E_S)$ by considering the equivalence class of $(g, \rho_W(g^{-1})(s)) \in G \times W$.

At each point $x \in M = G/G'$, let $(R_S)_x$ be the subspace of $J^p_x(E_S)$ defined by

$$(R_S)_x = \{\, j^p_x(\sigma_s) \mid s \in S \,\}.$$

Let R_S be the subbundle of $J^p(E_S)$ defined by

$$R_S = \bigcup_{x \in M} (R_S)_x.$$

Then we have

Proposition. (Proposition 2.4.1 [S].) *R_S is an integrable system of linear differential equations of finite type of order p of type S and every local solution of R_S is a restriction of σ_s for some $s \in S$.*

We call R_S the system of equations *modeled after* S. In the case when S is the typical symbol of type $(\mathfrak{l}, \rho)$, it follows from (2.5) that $G/G' = L/L'$. Moreover, when ρ is the irreducible representation of $\mathfrak{l}$ given in §1, we see that R^ρ coincides with the system of equations modeled after S.

2.4. Normal Reduction. Let R be an integrable system of linear differential equations of finite type of order p of type S on E. Then R is a vector bundle over the base manifold M with typical fibre S. A frame z of R at $x \in M$ is a linear isomorphism of S onto R_x. Let $F(R)$ be the frame bundle of R:

$$F(R) = \bigcup_{x \in M} F_x(R),$$

where $F_x(R)$ denotes the set of all frames of R at $x \in M$. $F(R)$ is a principal $GL(S)$-bundle over M. The flat connection ∇ in R induces the connection and the connection form $\tilde{\omega}$ on $F(R)$ is a $\mathfrak{gl}(S)$-valued 1-form. Se-ashi's theorem (Theorem A below) asserts the existence of a good reduction of the pair $(F(R), \tilde{\omega})$ for a system R with the typical symbol of type $(\mathfrak{l}, \rho)$. This reduction is carried out in several steps.

First, let $\{S^r\}_{r \geqq 0}$ be the filtration of S. The associated graded vector space $\mathrm{gr}(S) = \bigoplus_{r \geqq 0} S^r/S^{r+1}$ can be naturally identified with $S = \bigoplus_{r \geqq 0} S_r$. Let $GL^{(0)}(S)$ denote the subgroup of $GL(S)$ consisting of all elements $a \in GL(S)$ which preserve the filtration $\{S^r\}_{r \geqq 0}$ of S. For $a \in GL^{(0)}(S)$, we denote by $\mathrm{gr}(a) \in GL(S)$ the induced automorphism of the graded vector space $S = \bigoplus_{r=0}^{p} S_r$. Define

$$G^{(0)} = \{\, a \in GL^{(0)}(S) \mid gr(a) \in G_0 \,\}.$$

The Lie algebra of $G^{(0)}$ is given by $\mathfrak{g}^{(0)} = \mathfrak{g}_0 \oplus \bigoplus_{r=1}^{p-1} \mathfrak{gl}(S)_r$. Then we have the natural reduction of the structure group $GL(S)$ of $F(R)$ to $G^{(0)}$ as follows : At each $x \in M$, R_x has a filtration $\{R_x^r\}_{r \geqq 0}$ given by

$$R_x^r = \text{Ker } (\pi^p_{r-1} : R_x \to J_x^{r-1}(E))$$

Put

$$\widehat{P}_x(R) = \{ z \in F_x(R) \mid z(S^r) \subset R_x^r \quad \text{for any } r \}.$$

Obviously, $\widehat{P}(R) = \bigcup_{x \in M} \widehat{P}_x(R)$ is a principal
$GL^{(0)}(S)$-subbundle of $F(R)$. Since $\mathfrak{g}_r = R_r \cap (S^k T^* \otimes E)$ denotes the r-th symbol of R, each frame $z \in \widehat{P}_x(R)$ induces a graded map $\text{gr}(z) : S_r \to (\mathfrak{g}_r)_x$. We put

$$P_x(R) = \{ z \in \widehat{P}_x(R) \mid \text{gr}(z) \text{ is the extension of isomorphisms } V \cong T_x \text{ and } W \cong E_x \}.$$

Then $P(R) = \bigcup_{x \in M} P_x(R)$ is a principal $G^{(0)}$-subbundle of $F(R)$. Let $\pi : P(R) \to M$ be the bundle projection and let ω be the restriction to $P(R)$ of the connection form $\tilde{\omega}$ on $F(R)$. According to the decomposition $\mathfrak{gl}(S) = \bigoplus_{r=-p+1}^{p-1} \mathfrak{gl}(S)_r$, the form ω is decomposed as

$$\omega = \sum_r \omega_r.$$

It has the following properties (Proposition 3.2.2 [S]):

$$(2.6) \qquad \begin{cases} (1) \quad d\omega + \dfrac{1}{2}\omega \wedge \omega = 0, \\ (2) \quad \omega_r = 0 \quad \text{for } r \leqq -2, \\ (3) \quad \omega_{-1} \text{ is a } \mathfrak{g}_{-1}\text{-valued basic form, that is,} \\ \qquad \omega_{-1} \text{ gives the isomorphism } T_z(P(R))/\text{Ker } \pi \cong \mathfrak{g}_{-1} \text{ at each } z \in P(R). \end{cases}$$

The pair $(P(R), \omega)$ characterizes the equivalence class of the system R (Proposition 3.3.1 [S]). Namely, let R and R' be integrable systems of type S. Then an isomorphism ϕ of R onto R' induces the bundle isomorphism $P(\phi) : (P(R), \omega) \to (P(R'), \omega')$, i.e., $P(\phi)$ is a bundle isomorphism of $P(R)$ onto $P(R')$ satisfying $P(\phi)^*\omega' = \omega$. Conversely, for any isomorphism $\Psi : (P(R), \omega) \to (P(R'), \omega')$, there exists a unique isomorphism ϕ of R onto R' such that $\Psi = P(\phi)$.

Second, in order to state the normality condition for G'-reduction of $P(R)$, we prepare the Spencer cohomology associated with the adjoint representation of $\mathfrak{l}_{-1}$ on $\mathfrak{gl}(S)$.

On the space $C = \bigoplus C^{p,q}$ of cochains

$$C^{p,q} = \wedge^q(\mathfrak{l}_{-1})^* \otimes \mathfrak{gl}(S)_{p-1},$$

we define the coboundary operator $\partial : C^{p,q} \to C^{p-1,q+1}$ by

$$\partial c(X_0, \ldots, X_q) = \sum_j (-1)^j [\rho(X_j), c(X_o, \ldots, \hat{X}_j, \ldots, X_q)].$$

The cohomology group $H^q(\mathfrak{l}_{-1}, \mathfrak{gl}(S)) = \bigoplus_p H^{p,q}(\mathfrak{l}_{-1}, \mathfrak{gl}(S))$ of this cochain complex (C, ∂) is called the Spencer cohomology group associated with the adjoint representation of $\mathfrak{l}_{-1}$ on $\mathfrak{gl}(S)$. Moreover, the adjoint operator $\partial^* : C^{p-1,q+1} \to C^{p,q}$ is given by

$$\partial^* c(X_1, \ldots, X_q) = \sum_i [\rho(E^i), c(E_i, X_1, \ldots, X_q)],$$

where $\{E_i\}$ is a basis of $\mathfrak{l}_{-1}$ and $\{E^i\}$ is the dual basis of $\mathfrak{l}_1$ relative to the Killing form B. Let τ be the complex conjugation relative to a compact real form of $\mathfrak{l}$ such that $\tau(\mathfrak{l}_1) = \mathfrak{l}_{-1}$ and $\tau(\mathfrak{l}_0) = \mathfrak{l}_0$. We have a (hermitian) inner product given by $\{X, Y\} = -B(X, \tau(Y))$. Moreover, since $\mathfrak{l}$ is simple, we can find an inner product $\langle , \rangle$ on S such that $\langle \rho(X)(s), s' \rangle + \langle s, \rho(\tau(X))(s') \rangle = 0$ for $s, s' \in S$ and $X \in \mathfrak{l}$. Then we define the inner product $(,)$ on $\mathfrak{gl}(S)$ by $(u, v) =$trace (uv^*), where $u, v \in \mathfrak{gl}(S)$ and v^* is the adjoint of v relative to $\langle , \rangle$. These inner products induce naturally an inner product on $C^{p,q}$. Then, relative to this inner product, ∂^* is seen to be the adjoint of ∂. Thus we can develop a harmonic theory for (C, ∂), using the laplacian $\Delta = \partial\partial^* + \partial^*\partial$. In fact, we will apply the harmonic theory of Kostant to compute $H^{p,1}(\mathfrak{l}_{-1}, \mathfrak{g}^\perp)$ in §2.5. We denote by $\mathcal{H}$ the harmonic projection. For $\mathfrak{l}$-submodule $\mathfrak{g}^\perp$ of $\mathfrak{gl}(S)$, we put $C(\mathfrak{g}^\perp) = \wedge(\mathfrak{l}_{-1})^* \otimes \mathfrak{g}^\perp$. Then $(C(\mathfrak{g}^\perp), \partial)$ is a subcomplex of (C, ∂).

Let $(Q(R), \chi)$ be a G'-reduction of $(P(R), \omega)$; i.e., $Q(R)$ is a G'-principal subbundle of $P(R)$ and χ is the restriction of ω to $Q(R)$. According to the decomposition $\mathfrak{gl}(S) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, the form χ is decomposed as

$$\chi = \chi_{\mathfrak{g}} + \chi_{\mathfrak{g}^\perp}.$$

Since Tr is $Ad(G')$-invariant, we have $R_a^* \chi_{\mathfrak{g}} = \mathrm{Ad}(a^{-1})\, \chi_{\mathfrak{g}}$ and $R_a^* \chi_{\mathfrak{g}^\perp} = \mathrm{Ad}(a^{-1})\, \chi_{\mathfrak{g}^\perp}$ for any $a \in G'$. For $X \in \mathfrak{g}'$, $\chi_{\mathfrak{g}^\perp}(X^*) = 0$ since $\chi(X^*) = X$. From (2) and (3) of (2.6), we have $(\chi_{\mathfrak{g}^\perp})_p = 0$ for $p \leqq -1$. Moreover, $\chi_{\mathfrak{g}}$ gives an isomorphism between $T_u(Q(R))$ and $\mathfrak{g}$ at each point $u \in Q(R)$. Namely, we have (Proposition 5.1.1 [S]) the following.

(1) $(Q(R), \chi_{\mathfrak{g}})$ is a Cartan connection of type G/G' over M.

(2) $\chi_{\mathfrak{g}^\perp}$ is a tensorial 1-form on $Q(R)$.

We now define a $C^1(\mathfrak{g}^\perp)(= \mathrm{Hom}(\mathfrak{l}_{-1}, \mathfrak{g}^\perp))$-valued function c on $Q(R)$ by

$$c(u)(X) = \chi_{\mathfrak{g}^\perp}(X_u^*) \qquad \text{for } u \in Q(R), X \in \mathfrak{l}_{-1}.$$

c is called the *structure function* on $Q(R)$. For each p, c^p denotes the $C^{p,1}(\mathfrak{g}^\perp)$-component of c, i.e., $c^p(u)(X) = (\chi_{\mathfrak{g}^\perp})_{p-1}(X_u^*)$. Then

(2.7) $$c^p = 0 \qquad \text{for} \quad p \leqq 0.$$

We note here that, if c vanishes identically, we have $\chi = \chi_{\mathfrak{g}}$ and, from (1) of (2.6), $(Q(R), \chi)$ is a flat Cartan connection of type G/G'.

A G'-reduction $(Q(R), \chi)$ is said to be *normal* if the function c is ∂^*-closed. Now we can state Se-ashi's Theorem (Theorem 5.1.2, Theorem 5.2.2 [S]) as follows.

Theorem A. (1) *For every integrable system R of differential equations of type $(\mathfrak{l}, \rho)$, there exists a unique normal reduction $(Q(R), \chi)$ of $(P(R), \omega)$.*

(2) *Let R and R' be integrable systems of type $(\mathfrak{l}, \rho)$. Then an isomorphism ϕ of R onto R' induces the isomorphism $Q(\phi) : (Q(R), \chi) \to (Q(R'), \chi')$, i.e., $Q(\phi)$ is a bundle*

isomorphism of $Q(R)$ onto $Q(R')$ satisfying $Q(\phi)^*\chi' = \chi$. Conversely, for an isomorphism $\Psi : (Q(R), \chi) \to (Q(R'), \chi')$, there exists a unique isomorphism ϕ of R onto R' such that $\Psi = Q(\phi)$.

(3) *If the structure function c vanishes identically, then R is locally isomorphic with the model system of type $(\mathfrak{l}, \rho)$. Furthermore, the harmonic part $\mathcal{H}c$ of c gives a fundamental system of invariants of R, i.e., c vanishes if and only if $\mathcal{H}c$ vanishes.*

2.5. Vanishing theorem on $H^1(\mathfrak{l}_{-1}, \mathfrak{g}^\perp)$. Let us recall some facts on simple graded Lie algebras $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ of the first kind, following [Y], which are necessary in the subsequent discussion.

Let Z be the characteristic element of $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Since $\mathrm{ad}(Z)$ is a semi-simple endomorphism of $\mathfrak{l}$, we can take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{l}$ containing Z. Let Φ be the set of roots of $\mathfrak{l}$ relative to $\mathfrak{t}$. Then we have the root space decomposition of $\mathfrak{l}$:

$$\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{l}_\alpha,$$

where $\mathfrak{l}_\alpha = \{ X \in \mathfrak{l} \mid [H, X] = \alpha(H)X \quad \text{for all} \quad H \in \mathfrak{t} \}$ is the root space for $\alpha \in \Phi$. We have by definition $\alpha(Z) = -1, 0$ or 1 for any $\alpha \in \Phi$. Let us choose a simple root system $\Delta = \{ \alpha_1, \dots, \alpha_l \}$ of Φ such that $\alpha(Z) \geqq 0$ for all $\alpha \in \Delta$. Then there exists a unique simple root $\alpha_{i_0} \in \Delta$ such that $\alpha_{i_0}(Z) = 1$, $\alpha_i(Z) = 0$ for $i \neq i_0$ and the gradation is given by

$$\mathfrak{l}_0 = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_0^+} (\mathfrak{l}_\alpha \oplus \mathfrak{l}_{-\alpha}), \tag{2.8}$$

$$\mathfrak{l}_{-1} = \bigoplus_{\alpha \in \Phi_1^+} \mathfrak{l}_{-\alpha}, \ \mathfrak{l}_1 = \bigoplus_{\alpha \in \Phi_1^+} \mathfrak{l}_\alpha,$$

where $\Phi_p^+ = \{ \alpha \in \Phi^+ \mid \alpha(Z) = p \}$ and Φ^+ is the set of positive roots. Because of the partition $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$, we see that $n_{i_0}(\theta) = 1$ for the highest root $\theta = \sum_{i=1}^l n_i(\theta)\alpha_i$ and that

$$\Phi_p^+ = \{ \alpha = \sum_{i=1}^l n_i(\alpha)\alpha_i \in \Phi^+ \mid n_{i_0}(\alpha) = p \} \qquad \text{for} \quad p = 0, 1. \tag{2.9}$$

Conversely, let $\mathfrak{l}$ be a simple Lie algebra over $\mathbb{C}$. Let us fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{l}$ and a simple root system $\Delta = \{ \alpha_1, \dots, \alpha_l \}$ of Φ. Choose a simple root α_{i_0} such that $n_{i_0}(\theta) = 1$ for the highest root $\theta = \sum_{i=1}^l n_i(\theta)\alpha_i$, and define the partition $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$ by (2.9). Then we can construct the gradation of $\mathfrak{l}$ of the first kind by (2.8), i.e., by defining the characteristic element $Z \in \mathfrak{t}$ by

$$\alpha_i(Z) = \begin{cases} 1 & \text{if} \quad i = i_0, \\ 0 & \text{if} \quad i \neq i_0. \end{cases} \tag{2.10}$$

We denote the simple graded Lie algebra $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ obtained in this manner by $(X_l, \{\alpha_{i_0}\})$, when $\mathfrak{l}$ is a simple Lie algebra of type X_l. Here X_l stands for the Dynkin diagram of $\mathfrak{l}$ representing Δ and α_{i_0} is a vertex of X_l with the coefficient 1 for the highest root. It is known [Y, §3] that simple graded Lie algebras of the first kind are completely classified by the diagram automorphism of $(X_l, \{\alpha_{i_0}\})$. For example, the gradation of $\mathfrak{l} = \mathfrak{sl}(n-2, \mathbb{C})$ given in (1.1) corresponds to $(A_{n-3}, \{\alpha_{k-1}\})$. We refer the reader to [Y, §4.4] for the detail.

Let $\tau : \mathfrak{l} \to \mathfrak{gl}(T)$ be an irreducible representation with the highest weight Λ. Let t_Λ be a maximal vector in T of the highest weight Λ. Then an isotropy algebra at $[t_\Lambda] \in P(T)$ coincides with $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$ if and only if $(\Lambda, \alpha_{i_0}) \neq 0$ and $(\Lambda, \alpha_i) = 0$ for simple roots α_i other than α_{i_0}, where $(,)$ denotes the inner product in $(\mathfrak{t}_\mathbb{R})^*$.

Let $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$ be the dual representation of τ ; i.e., $S = T^*$ is the dual space of T and $\rho = \tau^*$ is defined by

$$\langle \rho(X)(\xi), t\rangle + \langle \xi, \tau(X)(t)\rangle = 0,$$

for $X \in \mathfrak{l}, t \in T, \xi \in T^*$ and $\langle\ ,\ \rangle$ is the canonical pairing between T^* and T. Then ρ is an irreducible representation with the lowest weight $\Gamma = -\Lambda$. Hence the minimum eigenvalue λ_0 of $\rho(Z)$ is given by $\lambda_0 = \Gamma(Z)$. From (2.10), we see that the eigenvalues of $\rho(Z)$ are of the form ; $\lambda_0, \lambda_0 + 1, \ldots, \lambda_0 + p_0 - 1 = \widehat{\Lambda}(Z)$, where $\widehat{\Lambda}$ is the highest weight of ρ. When $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is the isotropy algebra at $[t_\Lambda]$, the λ_0-eigenspace of $\rho(Z)$ coincides with the weight space for Γ, i.e., $S_0 = \langle s_1 \rangle$ in the notation of §1.

Given an irreducible representation $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$ on S, consider the adjoint representation $\mathrm{ad} \circ \rho : \mathfrak{l} \to \mathfrak{gl}(\mathfrak{gl}(S))$ on $\mathfrak{gl}(S)$. Then, from $[\rho(Z), Y](s) = \rho(Z)Y(s) - rY(s)$ for $s \in S_r$, we have

$$Y(S_r) \subset S_{l+r} \quad \text{for all } r \quad \text{if and only if} \quad [\rho(Z), Y] = lY.$$

Thus $\rho(Z) \in \mathfrak{gl}(S)$ is the characteristic element of the gradation of $\mathfrak{gl}(S) = \bigoplus_r \mathfrak{gl}(S)_r$.

To state the theorem of Kostant, we prepare the notation for the Weyl group W of the root system Φ. For an element $\sigma \in W$, we put $\Phi^- = -\Phi^+$ and $\Phi_\sigma = \sigma(\Phi^-) \cap \Phi^+$. Then $\sigma(\delta) = \delta - \langle \Phi_\sigma \rangle$, where $\delta = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ and $\langle \Phi_\sigma \rangle$ denotes the sum of all elements in Φ_σ. For a fixed $(X_l, \{\alpha_{i_0}\})$, we define the subset W^0 of W by putting

$$W^0 = \{\sigma \in W \mid \Phi_\sigma \subset \Phi_1^+\}.$$

Moreover, we put

$$W(q) = \{\sigma \in W \mid n(\sigma) = q\} \quad \text{and} \quad W^0(q) = W^0 \cap W(q),$$

where $n(\sigma)$ is the number of roots in Φ_σ. For an element $\sigma \in W^0(q)$, we put $x_{\Phi_\sigma} = x_{\beta_1} \wedge \cdots \wedge x_{\beta_q}$ where $\Phi_\sigma = \{\beta_1, \ldots, \beta_q\} \subset \Phi_1^+$ and x_{β_i} is a root vector for the root $\beta_i \in \Phi_1^+$.

The theorem due to Kostant that we utilize is the following.

Theorem B. (Proposition 10.1 [MM], Theorem (Kostant) [Y, §5.1].) *Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra over $\mathbb{C}$ represented by $(X_l, \{\alpha_{i_0}\})$ as above. Let $\tau : \mathfrak{l} \to \mathfrak{gl}(T)$ be an irreducible representation of $\mathfrak{l}$ on T with the lowest weight Γ.*

Then the harmonic space $\mathcal{H}^q$ of the cochain complex $C^q = T \otimes \wedge^q(\mathfrak{l}_{-1})^*$ can be decomposed into the irreducible $\mathfrak{l}_0$-module as follows:

$$\mathcal{H}^q = \bigoplus_{\sigma \in W^0(q)} \mathcal{H}^{\xi_\sigma},$$

where $\mathcal{H}^{\xi_\sigma}$ is the irreducible $\mathfrak{l}_0$-module with the lowest weight $\xi_\sigma = \sigma(\Gamma - \delta) + \delta = \sigma(\Gamma) + \langle \Phi_\sigma \rangle$ generated by the lowest weight vector

$$t_{\sigma(\Gamma)} \otimes x_{\Phi_\sigma},$$

where $t_{\sigma(\Gamma)}$ is a weight vector in T with weight $\sigma(\Gamma)$ and $x_{\Phi_\sigma} = x_{\beta_1} \wedge \cdots \wedge x_{\beta_q} \in \wedge^q \mathfrak{l}_1 \cong \wedge^q(\mathfrak{l}_{-1})^*$.

We apply this theorem to our case when $q = 1$. In this case we have $W^0(1) = \{\sigma_{i_0}\}$, where $\sigma_{i_0} = \sigma_{\alpha_{i_0}}$ is the reflection corresponding to the simple root α_{i_0}. Hence $\mathcal{H}^1$ is an irreducible $\mathfrak{l}_0$-module with the lowest weight $\xi_{i_0} = \sigma_{i_0}(\Gamma) + \alpha_{i_0}$.

Now we show the following vanishing theorem for $H^{p,1}(\mathfrak{l}_{-1}, \mathfrak{g}^\perp)$.

Theorem 2. *Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra over $\mathbb{C}$ and let $M = L/L'$ be the model space associated with $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Let $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$ be an irreducible representation on S and $H^1(\mathfrak{l}_{-1}, \mathfrak{g}^\perp)$ be the first Lie algebra cohomology associated with the adjoint representation of $\mathfrak{l}_{-1}$ on $\mathfrak{g}^\perp$ induced from $\mathrm{ad} \circ \rho : \mathfrak{l}_{-1} \to \mathfrak{gl}(\mathfrak{gl}(S))$, where $\mathfrak{gl}(S) = \mathfrak{g} \oplus \mathfrak{g}^\perp$.*

Then, for each $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$,

$$H^{p,1}(\mathfrak{l}_{-1}, \mathfrak{g}^\perp) = \{0\} \quad \text{for all } p \geqq 1,$$

except when M is a projective space $\mathbb{P}^m$ or a hyperquadric Q^m.

Proof. The adjoint representation $\mathrm{ad} \circ \rho : \mathfrak{l} \to \mathfrak{gl}(\mathfrak{gl}(S))$ on $\mathfrak{gl}(S)$ is decomposable according to the decomposition

$$\mathfrak{gl}(S) = \mathfrak{g} \oplus \mathfrak{g}^\perp,$$

and the gradation $\mathfrak{gl}(S) = \oplus_r \mathfrak{gl}(S)_r$ coincides with the eigenspace decomposition of $\mathrm{ad} \circ \rho(Z)$. To utilize Theorem B, we further decompose $\mathfrak{g}^\perp$ into direct sum of irreducible $\mathfrak{l}$-modules

$$\mathfrak{g}^\perp = \bigoplus m_\Gamma T_\Gamma,$$

where T_Γ is an irreducible $\mathfrak{l}$-submodule with the lowest weight Γ. Then we have

$$H^1(\mathfrak{l}_{-1}, \mathfrak{g}^\perp) = \bigoplus m_\Gamma H^1(\mathfrak{l}_{-1}, T_\Gamma).$$

By Theorem B, the harmonic space $\mathcal{H}^1_\Gamma$ representing $H^1(\mathfrak{l}_{-1}, T_\Gamma)$ is an irreducible $\mathfrak{l}_0$-module in $T_\Gamma \otimes \mathfrak{l}_1$ generated by

$$t_{\sigma_{i_0}(\Gamma)} \otimes x_{\alpha_{i_0}},$$

where $t_{\sigma_{i_0}(\Gamma)}$ is the weight vector with weight $\sigma_{i_0}(\Gamma)$ and $x_{\alpha_{i_0}}$ is a root vector for $\alpha_{i_0} \in \Phi_1^+$. Thus $\mathcal{H}^1_\Gamma \subset C^{p,1}(\mathfrak{g}^\perp)$, if $t_{\sigma_{i_0}(\Gamma)} \in \mathfrak{gl}(S)_{p-1}$. Hence p is given by

$$p - 1 = \sigma_{i_0}(\Gamma)(Z).$$

Let us compute the integer $\sigma_{i_0}(\Gamma)(Z)$. For each $\alpha \in \mathfrak{t}^*$, we denote by t_α and h_α the elements of $\mathfrak{t}$ defined by

$$B(t_\alpha, h) = \alpha(h) \quad \text{for } h \in \mathfrak{t} \qquad \text{and} \qquad h_\alpha = \frac{2t_\alpha}{(\alpha, \alpha)},$$

where $(\alpha, \alpha) = B(t_\alpha, t_\alpha)$ and B is the Killing form of $\mathfrak{l}$. Moreover, we put

$$\langle \mu, \alpha \rangle = \frac{2(\mu, \alpha)}{(\alpha, \alpha)} = \mu(h_\alpha) \qquad \text{for } \mu \in \mathfrak{t}^*.$$

Thus, for the simple root system $\{\alpha_1, \ldots, \alpha_l\}$ of Φ, $\{h_{\alpha_1}, \ldots, h_{\alpha_l}\}$ forms a basis of $\mathfrak{t}$. With respect to this basis, we put

$$Z = \sum_{i=1}^{l} a_i h_{\alpha_i}.$$

Then we compute

$$\begin{aligned} \sigma_{i_0}(\Gamma)(Z) =& (\Gamma - \langle \Gamma, \alpha_{i_0} \rangle \alpha_{i_o})(Z) = \Gamma(Z) - \langle \Gamma, \alpha_{i_0} \rangle \\ =& (a_{i_0} - 1)\langle \Gamma, \alpha_{i_0} \rangle + \sum_{i \neq i_0} a_i \langle \Gamma, \alpha_i \rangle \end{aligned} \tag{2.11}$$

Since Γ is the lowest weight, we have $\langle \Gamma, \alpha_i \rangle \leqq 0$ for $i = 1, \ldots, l$ and $\langle \Gamma, \alpha_j \rangle < 0$ for some j. Let us now check the sign of $(a_{i_0} - 1)$ and a_i. From (2.10), we have

$$\alpha_i(Z) = \sum_{j=1}^{l} \langle \alpha_i, \alpha_j \rangle a_j = \begin{cases} 1 & \text{if} \quad i = i_0, \\ 0 & \text{if} \quad i \neq i_0. \end{cases}$$

Hence, we see that $(a_1, \ldots, a_l)$ coincides with the i_0-th column vector of the inverse matrix C^{-1} of the Cartan matrix $C = (\langle \alpha_i, \alpha_j \rangle)$ of $\mathfrak{l}$. It is a well-known fact that all entries of C^{-1} are positive numbers (see, e.g., Table 1 [Hu, p.69]). Moreover, if $a_{i_0} > 1$, we see, from (2.11), that $\sigma_{i_0}(\Gamma)(Z) < 0$ for every Γ, i.e., $p < 1$ for every $\mathcal{H}^1_\Gamma \subset C^{p,1}(\mathfrak{g}^\perp)$. Hence we get $H^{p,1}(\mathfrak{l}_{-1}, \mathfrak{g}^\perp) = \{0\}$ for all $p \geqq 1$ in this case. Thus our task is to list up those $(X_l, \{\alpha_{i_0}\})$ for which $a_{i_0} \leqq 1$. In fact, from Table 1 [Hu, p.69], we obtain the following list of $(X_l, \{\alpha_{i_0}\})$ for which $a_{i_0} \leqq 1$:

$$\begin{array}{llllll} (A_l, \{\alpha_1\}) & a_1 = \dfrac{l}{l+1} & (l \geqq 1), & (A_3, \{\alpha_2\}) & a_2 = 1 & \\ (B_l, \{\alpha_1\}) & a_1 = 1 & (\mathfrak{l} \geqq 2), & (D_l, \{\alpha_1\}) & a_1 = 1 & (\mathfrak{l} \geqq 4), \end{array}$$

Here we identify $(B_2, \{\alpha_1\}) \cong (C_2, \{\alpha_2\})$, $(D_4, \{\alpha_1\}) \cong (D_4, \{\alpha_3\}) \cong (D_4, \{\alpha_4\})$ and $(A_l, \{\alpha_1\}) \cong (A_l, \{\alpha_l\})$ by diagram automorphisms. One can easily check (cf. [Y, §4.4]) that, when $(X_l, \{\alpha_{i_0}\})$ coincides with one of the above list, the model space $M = L/L'$ corresponds to $\mathbb{P}^l$ $(l \geqq 1)$, $Q^4 = Gr_{2,4}$, Q^{2l-1} $(l \geqq 2)$ and $Q^{2(l-1)}$ $(l \geqq 4)$. This completes the proof of Theorem C.

Now, combining Theorem A (3), Theorem C and (2.7), we obtain

Corollary 3. *Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra over $\mathbb{C}$ and let $M = L/L'$ be the model space associated with $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Let $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$ be an irreducible representation of $\mathfrak{l}$. Then, except when $M = \mathbb{P}^m$ or Q^m, every integrable system R of differential equations of type $(\mathfrak{l}, \rho)$ is locally isomorphic with the model system R^ρ of type $(\mathfrak{l}, \rho)$.*

3. Proof of Theorem 1

In this section we will show the inequivalence of $E(k,n)$ and $R(k,n)$ for $(k,n) \neq (3,6)$ and prove Theorem. Recall that $R(k,n)$ is the model system of type $(\mathfrak{l}, \rho_0)$, where $\mathfrak{l} = \mathfrak{sl}(n-s, \mathbb{C})$ with the gradation $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ given by (1.1) and ρ_0 is the exterior representation of $\mathfrak{sl}(n-2, \mathbb{C})$ on $\bigwedge^{n-k-1} \mathbb{C}^{n-2}$. By the argument in §2.2 and §2.3, we see that $R(k,n)$ is an integrable system of order $p_0 = \min\{k, n-k\}$ over the model space $M = Gr_{k-1,n-2}$. Hence, by Corollary D, $R(k,n)$ is characterized solely by its symbol. Thus, to prove Theorem, we need only to show that $E(k,n)$ is not of type $(\mathfrak{l}, \rho_0)$ for $(k,n) \neq (3,6)$, i.e., the symbol of $E(k,n)$ at a generic point is not equivalent to the typical symbol of $R(k,n)$ discussed in §2.2 for $(k,n) \neq (3,6)$.

3.1. The symbol of the Plücker embedding

We recall the calculations in §2.2. Let us take the following basis for $V = \mathfrak{l}_{-1}$ and S_r,

$$V = \mathfrak{l}_{-1} = \langle E_{pi} \mid 1 \leqq i \leqq k-1, k \leqq p \leqq n-2 \rangle,$$

$$S_r = \langle s(i_1, \ldots, i_r, p_1, \ldots, p_r) \mid 1 \leqq i_1 < \cdots < i_r \leqq k-1, k \leqq p_1 < \cdots < p_r \leqq n-2 \rangle,$$

where

$$s(i_1, \ldots, i_r, p_1, \ldots, p_r) = e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e(p_1, \ldots, p_r) \in S = \wedge^{n-k-1}\mathbb{C}^{n-2}.$$

Then we have

$$\iota_r(s(i_1, \ldots, i_r, p_1, \ldots, p_r))(X, \ldots, X) = \\ r!(-1)^r (\sum_\sigma \operatorname{sgn}\sigma \, X_{i_1 p_{\sigma(1)}} \cdots X_{i_r p_{\sigma(r)}}) \, e_{p_1} \wedge \cdots \wedge e_{p_r} \wedge e(p_1, \ldots, p_r),$$

for $X = \sum_{ip} X_{ip} E_{pi} \in V$. Thus, by fixing a basis of $W = S_0$ and identifying SV^* with the ring of polynomials on V, we see that $S_1 = V^*$ and $S_r \subset S^r V^*$ is spanned by the minor determinants of degree r of the matrix (X_{ip}). By construction of $R(k,n)$,

$$S = \bigoplus_{r=0}^{p_0} S_r$$

is the typical symbol of $R(k,n)$. Hence, putting $R_r(k,n) = \pi^{p_0}_r(R(k,n))$, the symbol $\mathfrak{g}_r = R_r(k,n) \cap (S^r T^* \otimes E)$ of $R_r(k,n)$ is of type $S_r \subset S^r V^*$ at each point of $M = Gr_{k-1,n-2}$.

Now let us first show that $R(k,n)$ is essentially a second order system. More precisely, we claim

$$R(k,n) \textit{ is the } (p_0-2)\textit{-th prolongation of } R_2(k,n)$$

Namely p_0-th order system $R(k,n)$ is obtained from the second order system $R_2(k,n)$ by adding successive (partial) derivatives to $R_2(k,n)$. In order to show this, since $\pi^r_{r-1} : R_r(k,n) \to R_{r-1}(k,n)$ is onto by construction, we need only to show that the symbol $\mathfrak{g}_r$ of $R_r(k,n)$ is the $(r-2)$-th prolongation of $\mathfrak{g}_2$. In fact we have

Lemma 3.1. *The space $S_r \subset S^r V^*$ is equal to the $(r-2)$-th prolongation $p^{(r-2)}(S_2)$ of $S_2 \subset S^2 V^*$.*

Here we recall that s-th (algebraic) prolongation $p^{(s)}(S_2)$ of S_2 is given by

$$p^{(s)}(S_2) = S_2 \otimes \otimes^s V^* \cap S^{s+2} V^*.$$

Proof. Let T_r be the annihilator of S_r in $S^r V$, where we identify $S^r V$ with the dual space of $S^r V^*$. Then T_2 is generated by the following vectors ;

$$\begin{aligned} E_{pi} \cdot E_{qj} + E_{qi} \cdot E_{pj} \qquad & (1 \leqq i < j \leqq k-1, k \leqq p < q \leqq n-2) \\ E_{pj} \cdot E_{qj} \qquad & (1 \leqq i \leqq k-1, k \leqq p < q \leqq n-2) \\ E_{qi} \cdot E_{qj} \qquad & (1 \leqq i < j \leqq k-1, k \leqq q \leqq n-2) \\ E_{qj}^2 \qquad & (1 \leqq j \leqq k-1, k \leqq q \leqq n-2) \end{aligned}$$

where $\cdot$ denotes the symmetric product. Let $T_2^{(s)}$ denote the annihilator of $p^{(s)}(S_2)$ in $S^{s+2}V$. Then we have

$$T_2^{(s)} = \langle f \cdot g \mid f \in S^s V \quad \text{and} \quad g \in T_2 \rangle.$$

Moreover, since S_{s+2} is generated by the minor determinants of degree $s+2$ of the matrix (X_{ip}), we have

$$T_2^{(s)} \subset T_{s+2}. \tag{3.1}$$

We observe here that each monomial $E_{p_1 i_1} \cdot E_{p_2 i_2} \cdot \cdots \cdot E_{p_{s+2} i_{s+2}}$ in $S^{s+2}V$ belongs to $T_2^{(s)}$ if there is a repetition among the indices $i_1, \ldots, i_{s+2}$ or $p_1, \ldots, p_{s+2}$. On the other hand, given indices $i_1, \ldots, i_{s+2}$ and $p_1, \ldots, p_{s+2}$ such that $1 \leqq i_1 < \cdots < i_{s+2} \leqq k-1$ and $k \leqq p_1 < \cdots < p_{s+2} \leqq n-2$, we see that $(s+2)!$ monomials

$$E_{p_1 i_{\sigma(1)}} \cdot E_{p_2 i_{\sigma(2)}} \cdot \cdots \cdot E_{p_{s+2} i_{\sigma(s+2)}},$$

where σ runs for all permutations of degree $s+2$, span (at most) 1-dimensional subspace modulo $T_2^{(s)}$. In fact, to see this, it is enough to line up all the permutations of degree $(s+2)$ in one row so that each permutation $(l_1, \ldots, l_{s+2})$, where $l_i = \sigma(i)$ $(i = 1, 2, \ldots, s+2)$, is obtained by a transposition from the former permutation in this row. Then the dimension count shows

$$\operatorname{codim} T_2^{(s)} \leqq \binom{k-1}{s+2} \times \binom{n-k-1}{s+2} = \dim S_{s+2},$$

which, together with (3.1), implies $T_2^{(s)} = T_{s+2}$. This completes the proof of Lemma.

In view of this lemma, we will discuss the inequivalence of second order systems $E(k,n)$ and $R_2(k,n)$ in §3.3. Here the symbol $\mathfrak{g}_2 = R_2(k,n) \cap (S^2T^* \otimes E)$ of $R_2(k,n)$ is of type $S_2 \subset S^2V^*$ at each point of $M = Gr_{k-1,n-2}$. Let $\{e_{ip}\}$ denote the dual basis of $\{E_{pi}\}$ in V^*. Then recall that $S_2 \subset S^2V^*$ is generated by the following elements of S^2V^* ;

$$S_{ijpq} = e_{ip} \cdot e_{jq} - e_{iq} \cdot e_{jp}, \qquad (1 \leqq i < j \leqq k-1, k \leqq p < q \leqq n-2).$$

3.2. The symbol of $E(k,n)$

For a set of parameters

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad \sum_{j=1}^{n} \alpha_j = n - k,$$

the hypergeometric system of type (k,n) is the system of linear differential equations:

$$\sum_{j=1}^{n} x_{lj} \frac{\partial u}{\partial x_{ij}} + \delta_{il} u = 0,$$

$$\sum_{i=1}^{k} x_{ij} \frac{\partial u}{\partial x_{ij}} - (\alpha_j - 1) u = 0,$$

$$\frac{\partial^2 u}{\partial x_{ip} \partial x_{jq}} - \frac{\partial^2 u}{\partial x_{iq} \partial x_{jp}} = 0,$$

where

$$(x_{ij}) \in M^*(k,n) = \{k \times n\text{-matrices such that no } k\text{-minor vanishes}\}.$$

The configuration space $X(k,n)$ of distinct n points on the projective $(k-1)$-space is by definition given as

$$X(k,n) = GL(k)\backslash M^*(k,n)/H(n),$$

where $H(n)$ is the group consisting of diagonal non-singular n-matrices. Though the above system is not defined on $X(k,n)$, its projective solutions are defined on it. So instead of transforming the system into a $GL(k) \times H(n)$-invariant form, we restrict this system to the "subset" of $M^*(k,n)$ defined as follows:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 0 & 1 & x_{2\ k+2} & \cdots & x_{2n} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 1 & x_{k\ k+2} & \cdots & x_{kn} \end{pmatrix}.$$

Note that any element of $M^*(k,n)$ can be taken to this form by $GL(k) \times H(n)$, in other words, this is a section of the projection $M^*(k,n) \to X(k,n)$. So in the following, we identify this subset with $X(k,n)$, i.e., we regard $(x_{ip}) \in X(k,n)$.

The restricted system $E(k,n) = E(k,n;\alpha_1,\dots,\alpha_n)$ consists of the following differential equations relative to the variables x_{ip}, $2 \leqq i \leqq k$, $k+2 \leqq p \leqq n$.

$$(3.2)\qquad \begin{aligned} (\alpha - 1 + \theta)\theta_{jq}u &= x_{jq}(\theta^q - \alpha_q + 1)(\theta_j + \alpha_j)u, \\ x_{jp}(\theta^p - \alpha_p + 1)\theta_{jq}u &= x_{jq}(\theta^q - \alpha_q + 1)\theta_{jp}u, \\ x_{iq}(\theta_i + \alpha_i)\theta_{jq}u &= x_{jq}(\theta_j + \alpha_j)\theta_{iq}u, \\ x_{iq}x_{jp}\theta_{ip}\theta_{jq}u &= x_{ip}x_{jq}\theta_{iq}\theta_{jp}u, \end{aligned}$$

where

$$\theta_{ip} = x_{ip}\frac{\partial}{\partial x_{ip}}, \quad \theta_i = \sum_{p=k+2}^{n} \theta_{ip}, \quad \theta^p = \sum_{i=2}^{k} \theta_{ip}, \quad \theta = \sum_{i=2}^{k}\sum_{p=k+2}^{n} \theta_{ip}.$$

and

$$\alpha = \alpha_2 + \cdots + \alpha_{k+1}.$$

Refer to [MSY1]. Here and in the following, the indices i and j run from 2 to k, and the indices p and q from $k+2$ to n.

Now let us calculate the symbol of $E(k,n)$. In the spirit of §2, we regard $E(k,n)$ as the subbundle of $J^2(E)$ defined by (3.2), where $E = \mathbb{C} \times X(k,n)$ is the trivial line bundle over the configuration space $X(k,n)$. Let $S_2(x) = E(k,n) \cap (S^2T_x^* \otimes \mathbb{C})$ be the symbol of $E(k,n)$ at $x = (x_{ip}) \in X(k,n)$. We regard $S_2(x)$ as a subspace of $S^2T_x^*$. Then, from (3.2), we see that the annihilator $T_2(x)$ of $S_2(x)$ in S^2T_x is generated by the following elements:

$$\begin{aligned} A_{jq} &= \sum_{i,p}(x_{ip}\xi_{ip}x_{jq}\xi_{jq} - x_{jq}x_{iq}\xi_{iq}x_{jp}\xi_{jp}), \\ B_{jpq} &= x_{jp}\sum_i x_{ip}\xi_{ip}x_{jq}\xi_{jq} - x_{jq}\sum_i x_{iq}\xi_{iq}x_{jp}\xi_{jp}, \\ C_{ijq} &= x_{iq}\sum_p x_{ip}\xi_{ip}x_{jq}\xi_{jq} - x_{jq}\sum_p x_{iq}\xi_{iq}x_{jp}\xi_{jp}, \\ D_{ijpq} &= x_{iq}x_{jp}x_{ip}\xi_{ip}x_{jq}\xi_{jq} - x_{ip}x_{jq}x_{iq}\xi_{iq}x_{jp}\xi_{jp}. \end{aligned}$$

where we put $\xi_{ip} = \frac{\partial}{\partial x_{ip}}$, and $\{\xi_{ip}\}$ forms a basis of T_x. Since

$$B_{jpq} = x_{jp}x_{jq}((\sum_i x_{ip}\xi_{ip})\xi_{jq} - (\sum_i x_{iq}\xi_{iq})\xi_{jp}),$$
$$C_{ijq} = x_{iq}x_{jq}((\sum_p x_{ip}\xi_{ip})\xi_{jq} - (\sum_p x_{jp}\xi_{jp})\xi_{iq}),$$
$$D_{ijpq} = x_{ip}x_{iq}x_{jp}x_{jq}(\xi_{ip}\xi_{jq} - \xi_{iq}\xi_{jp}),$$

and

$$A_{jq} \equiv x_{jq}(\sum_p x_{jp}\xi_{jp})(\sum_i (1-x_{iq})\xi_{iq}) \quad \text{modulo} \quad D_{ijpq},$$

$T_2(x)$ is generated by

$$A'_{jq} = \eta_j \eta^q,$$
$$B'_{jpq} = \eta^p \xi_{jq} - \eta^q \xi_{jp},$$
$$C'_{ijq} = \eta_i \xi_{jq} - \eta_j \xi_{iq},$$
$$D'_{ijpq} = \xi_{ip}\xi_{jq} - \xi_{iq}\xi_{jp},$$

where

$$\eta_j = \sum_p x_{jp}\xi_{jp}, \quad \eta^q = \sum_i (1-x_{iq})\xi_{iq}.$$

Furthermore, the first three are equal to the following, respectively, modulo the generator D'_{ijpq}.

$$\hat{A}_{jq} = (\sum_{i,p}(x_{ip} - x_{iq}x_{jp})\xi_{ip})\xi_{jq},$$
$$\hat{B}_{jpq} = (\sum_i (x_{iq} - x_{ip})\xi_{ip})\xi_{jq},$$
$$\hat{C}_{ijq} = (\sum_p (x_{ip} - x_{jp})\xi_{ip})\xi_{jq}.$$

Let us now compute the generators of $S_2(x)$. We denote by $\{e_{ip}\}$ the dual basis of $\{\xi_{ip}\}$. Since any elements of $S_2(x)$ are annihilated by above elements of $T_2(x)$, we look for the elements of the form

$$E_{ijpq} = e_{ip}\cdot e_{jq} + e_{iq}\cdot e_{jp} + \sum_{\ell<m,s} P^{ijpq}_{\ell ms} e_{\ell s}\cdot e_{ms} + \sum_{m,r<s} Q^{ijpq}_{mrs} e_{mr}\cdot e_{ms} + \sum_{m,s} R^{ijpq}_{ms} {e_{ms}}^2.$$

Obviously, this satisfies $D'_{\ell mrs}(E_{ijpq}) = 0$. By requiring E_{ijpq} to be annihilated by $\hat{C}_{\ell ms}$ and by $\hat{B}_{mrs}$, we can determine the coefficients P's and Q's as follows:

$$E_{ijpq} = e_{ip}\cdot e_{jq} + e_{iq}\cdot e_{jp} - \frac{x_{iq}-x_{jq}}{x_{ip}-x_{jp}} e_{ip}\cdot e_{jp} - \frac{x_{ip}-x_{jp}}{x_{iq}-x_{jq}} e_{iq}\cdot e_{jq} - \frac{x_{jq}-x_{jp}}{x_{iq}-x_{ip}} e_{ip}\cdot e_{iq} - \frac{x_{iq}-x_{ip}}{x_{jq}-x_{jp}} e_{jp}\cdot e_{jq} + \sum_{m,s} R^{ijpq}_{ms} {e_{ms}}^2.$$

The condition $\hat{A}_{ms}(E_{ijpq}) = 0$ is a little complicated; a calculation shows

$$\begin{aligned}
R_{ip}^{ijpq} &= -\frac{x_{jq} - x_{jp}x_{iq}}{(1-x_{ip})x_{ip}} + \frac{x_{jp}}{x_{ip}}\frac{x_{iq} - x_{jq}}{x_{ip} - x_{jp}} + \frac{x_{iq}}{x_{ip}}\frac{x_{jq} - x_{jp}}{x_{iq} - x_{ip}}, \\
R_{iq}^{ijpq} &= -\frac{x_{jp} - x_{jq}x_{ip}}{(1-x_{iq})x_{iq}} + \frac{x_{jq}}{x_{iq}}\frac{x_{ip} - x_{jp}}{x_{iq} - x_{jq}} + \frac{x_{ip}}{x_{iq}}\frac{x_{jq} - x_{jp}}{x_{iq} - x_{ip}}, \\
R_{jp}^{ijpq} &= -\frac{x_{iq} - x_{ip}x_{jq}}{(1-x_{jp})x_{jp}} + \frac{x_{ip}}{x_{jp}}\frac{x_{iq} - x_{jq}}{x_{ip} - x_{jp}} + \frac{x_{jq}}{x_{jp}}\frac{x_{iq} - x_{ip}}{x_{jq} - x_{jp}}, \\
R_{jq}^{ijpq} &= -\frac{x_{ip} - x_{iq}x_{jp}}{(1-x_{jq})x_{jq}} + \frac{x_{iq}}{x_{jq}}\frac{x_{ip} - x_{jp}}{x_{iq} - x_{jq}} + \frac{x_{jp}}{x_{jq}}\frac{x_{iq} - x_{ip}}{x_{jq} - x_{jp}}, \\
R_{ms}^{ijpq} &= 0 \quad \text{otherwise.}
\end{aligned}$$

We put

$$R_{ip} = R_{ip}^{ijpq};$$

then, we see that

$$\begin{aligned}
E_{ijpq} = e_{ip}\cdot e_{jq} + e_{iq}\cdot e_{jp} &- \frac{x_{iq} - x_{jq}}{x_{ip} - x_{jp}} e_{ip}\cdot e_{jp} - \frac{x_{ip} - x_{jp}}{x_{iq} - x_{jq}} e_{iq}\cdot e_{jq} \\
&- \frac{x_{jq} - x_{jp}}{x_{iq} - x_{ip}} e_{ip}\cdot e_{iq} - \frac{x_{iq} - x_{ip}}{x_{jq} - x_{jp}} e_{jp}\cdot e_{jq} \\
&+ R_{ip}{e_{ip}}^2 + R_{iq}{e_{iq}}^2 + R_{jp}{e_{jp}}^2 + R_{jq}{e_{jq}}^2.
\end{aligned} \tag{3.3}$$

Here we note that E_{ijpq} is a quadratic polynomial in four variables e_{ip}, e_{iq}, e_{jp} and e_{jq}. Thus, the space $S_2(x)$ is generated by these elements E_{ijpq} $\quad (2 \leqq i < j \leqq k, k+2 \leqq p < q \leqq n)$.

In the following, we use R_{ip} written in the form

$$\begin{aligned}
R_{ip} =& \frac{x_{iq}x_{jp} - x_{iq} - x_{jp} + x_{jq}}{x_{ip}} + \frac{x_{iq}x_{jp} - x_{jq}}{1 - x_{ip}} \\
&+ \frac{x_{iq} - x_{jq}}{x_{ip} - x_{jp}} + \frac{x_{jp} - x_{jq}}{x_{ip} - x_{iq}}.
\end{aligned} \tag{3.4}$$

3.3. Proof

By summarizing the discussion in the above subsections, our task is now to show the inequivalence of the symbol spaces $S_2(x)$ and S_2 for a generic point x of $X(k,n)$. More precisely, we need to show that, for a generic point $x \in X(k,n)$, there does not exist a linear isomorphism ϕ of V onto T_x such that $\phi^* : S^2T_x^* \to S^2V^*$ sends $S_2(x)$ onto S_2. In other words our task is to show, for a generic point $x \in X(k,n)$, the projective inequivalence of the varieties $V(S_2(x))$ and $V(S_2)$, where $V(S_2(x))$ and $V(S_2)$ are varieties in the projective spaces PT_x^* and PV^*, which are defined by the quadratic generators of $S_2(x)$ and S_2, respectively.

Here we note that, since the generators of S_2 are minor determinants of degree 2 of the matrix (e_{ip}), $V(S_2)$ is called the *Segre variety* and coincides with the image of $\mathbb{P}^{k-2} \times \mathbb{P}^{n-k-2}$ under the *Segre embedding*. Especially, we see that $V(S_2)$ is a smooth projective variety of dimension $n-4$. Referring to this fact, we will check the above inequivalence by looking at the most primitive invariants of varieties, i.e., by counting the dimension of $V(S_2(x))$. In fact we can check that

$$\dim V(S(x)) < n - 4,$$

at a generic point $x = (x_{ip}) \in X(k,n)$ as in the following.

Let us first examine the typical and easiest case when $(k,n) = (3,7)$. The dimension of S_2 is 3; the space S_2 is generated by

$$E_{2356},\ E_{2357},\ E_{2367}.$$

For ease of reference we index the coordinates as follows:

$$\begin{pmatrix} x_{25} & x_{26} & x_{27} \\ x_{35} & x_{36} & x_{37} \end{pmatrix} = \begin{pmatrix} x & \alpha_1 & \alpha_2 \\ \beta & \gamma_1 & \gamma_2 \end{pmatrix}.$$

Each E_* is a homogeneous polynomial of e_{ip}. We introduce inhomogeneous coordinates by

$$Y_1 = e_{26}/e_{25},\ Y_2 = e_{27}/e_{25},\ Z = e_{35}/e_{25},\ W_1 = e_{36}/e_{25},\ W_2 = e_{37}/e_{25}.$$

Then the E_*'s, more precisely the quotients $E_*/{e_{25}}^2$, are functions of the inhomogeneous coordinates. The explicit forms are given by (3.3) and (3.4) as follows:

$$\begin{aligned} E_{2356} =& W_1 + Y_1 Z - \frac{\alpha_1 - \gamma_1}{x - \beta} Z - \frac{x - \beta}{\alpha_1 - \gamma_1} Y_1 W_1 - \frac{\gamma_1 - \beta}{\alpha_1 - x} Y_1 - \frac{\alpha_1 - x}{\gamma_1 - \beta} Z W_1 \\ &+ A_1 + B_1 {Y_1}^2 + C_1 Z^2 + D_1 {W_1}^2, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \frac{\alpha_1\beta - \alpha_1 - \beta + \gamma_1}{x} + \frac{\alpha_1\beta - \gamma_1}{1 - x} + \frac{\alpha_1 - \gamma_1}{x - \beta} + \frac{\beta - \gamma_1}{x - \alpha_1}, \\ B_1 &= \frac{x\gamma_1 - x - \gamma_1 + \beta}{\alpha_1} + \frac{x\gamma_1 - \beta}{1 - \alpha_1} + \frac{x - \beta}{\alpha_1 - \gamma_1} + \frac{\gamma_1 - \beta}{\alpha_1 - x}, \\ C_1 &= \frac{\gamma_1 x - \gamma_1 - x + \alpha_1}{\beta} + \frac{\gamma_1 x - \alpha_1}{1 - \beta} + \frac{\gamma_1 - \alpha_1}{\beta - x} + \frac{x - \alpha_1}{\beta - \gamma_1}, \\ D_1 &= \frac{\beta\alpha_1 - \beta - \alpha_1 + x}{\gamma_1} + \frac{\beta\alpha_1 - x}{1 - \gamma_1} + \frac{\beta - x}{\gamma_1 - \alpha_1} + \frac{\alpha_1 - x}{\gamma_1 - \beta}; \end{aligned}$$

$$\begin{aligned} E_{2357} =& W_2 + Y_2 Z - \frac{\alpha_2 - \gamma_2}{x - \beta} Z - \frac{x - \beta}{\alpha_2 - \gamma_2} Y_2 W_2 - \frac{\gamma_2 - \beta}{\alpha_2 - x} Y_2 - \frac{\alpha_2 - x}{\gamma_2 - \beta} Z W_2 \\ &+ A_2 + B_2 {Y_2}^2 + C_2 Z^2 + D_2 {W_2}^2, \end{aligned}$$

where

$$\begin{aligned} A_2 &= \frac{\alpha_2\beta - \alpha_2 - \beta + \gamma_2}{x} + \frac{\alpha_2\beta - \gamma_2}{1 - x} + \frac{\alpha_2 - \gamma_2}{x - \beta} + \frac{\beta - \gamma_2}{x - \alpha_2}, \\ B_2 &= \frac{x\gamma_2 - x - \gamma_2 + \beta}{\alpha_2} + \frac{x\gamma_2 - \beta}{1 - \alpha_2} + \frac{x - \beta}{\alpha_2 - \gamma_2} + \frac{\gamma_2 - \beta}{\alpha_2 - x}, \\ C_2 &= \frac{\gamma_2 x - \gamma_2 - x + \alpha_2}{\beta} + \frac{\gamma_2 x - \alpha_2}{1 - \beta} + \frac{\gamma_2 - \alpha_2}{\beta - x} + \frac{x - \alpha_2}{\beta - \gamma_2}, \\ D_2 &= \frac{\beta\alpha_2 - \beta - \alpha_2 + x}{\gamma_2} + \frac{\beta\alpha_2 - x}{1 - \gamma_2} + \frac{\beta - x}{\gamma_2 - \alpha_2} + \frac{\alpha_2 - x}{\gamma_2 - \beta}; \end{aligned}$$

$$\begin{aligned} E_{2367} =& Y_1 W_2 + Y_2 W_1 - \frac{\alpha_2 - \gamma_2}{\alpha_1 - \gamma_1} Y_1 W_1 - \frac{\alpha_1 - \gamma_1}{\alpha_2 - \gamma_2} Y_2 W_2 - \frac{\gamma_2 - \gamma_1}{\alpha_2 - \alpha_1} Y_1 Y_2 - \frac{\alpha_2 - \alpha_1}{\gamma_2 - \gamma_1} W_1 W_2 \\ &+ A {Y_1}^2 + B {Y_2}^2 + C {W_1}^2 + D {W_2}^2, \end{aligned}$$

where

$$A=\frac{\alpha_2\gamma_1-\alpha_2-\gamma_1+\gamma_2}{\alpha_1}+\frac{\alpha_2\gamma_1-\gamma_2}{1-\alpha_1}+\frac{\alpha_2-\gamma_2}{\alpha_1-\gamma_1}+\frac{\gamma_1-\gamma_2}{\alpha_1-\alpha_2},$$
$$B=\frac{\alpha_1\gamma_2-\alpha_1-\gamma_2+\gamma_1}{\alpha_2}+\frac{\alpha_1\gamma_2-\gamma_1}{1-\alpha_2}+\frac{\alpha_1-\gamma_1}{\alpha_2-\gamma_2}+\frac{\gamma_2-\gamma_1}{\alpha_2-\alpha_1},$$
$$C=\frac{\gamma_2\alpha_1-\gamma_2-\alpha_1+\alpha_2}{\gamma_1}+\frac{\gamma_2\alpha_1-\alpha_2}{1-\gamma_1}+\frac{\gamma_2-\alpha_2}{\gamma_1-\alpha_1}+\frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2},$$
$$D=\frac{\gamma_1\alpha_2-\gamma_1-\alpha_2+\alpha_1}{\gamma_2}+\frac{\gamma_1\alpha_2-\alpha_1}{1-\gamma_2}+\frac{\gamma_1-\alpha_1}{\gamma_2-\alpha_2}+\frac{\alpha_2-\alpha_1}{\gamma_2-\gamma_1}.$$

Thus, on the Zariski open subset ($D_1\neq 0$ and $D_2\neq 0$) of $X(3,7)$, from the equations $E_{2356}=E_{2357}=0$, we can solve W_1 and W_2 in terms of Y_1, Z and Y_2, Z, respectively. Substituting these into $E_{2367}=0$, we get a non-trivial equation for Y_1,Y_2 and Z. Thus we see that $\dim V(S_2(x))=2$ at a generic point of $X(3,7)$, whereas $\dim V(S_2)=3$. More precisely, we observe this fact from the following computation of the differentials:

$$\begin{aligned}
dE_{2356}=&\left(1-\frac{x-\beta}{\alpha_1-\gamma_1}Y_1-\frac{\alpha_1-x}{\gamma_1-\beta}Z+2D_1W_1\right)dW_1\\
&+\left(Z-\frac{x-\beta}{\alpha_1-\gamma_1}W_1-\frac{\gamma_1-\beta}{\alpha_1-x}+2B_1Y_1\right)dY_1\\
&+\left(Y_1-\frac{\alpha_1-\gamma_1}{x-\beta}-\frac{\alpha_1-x}{\gamma_1-\beta}W_1+2C_1Z\right)dZ,\\
dE_{2357}=&\left(1-\frac{x-\beta}{\alpha_2-\gamma_2}Y_2-\frac{\alpha_2-x}{\gamma_2-\beta}Z+2D_2W_2\right)dW_2\\
&+\left(Z-\frac{x-\beta}{\alpha_2-\gamma_2}W_2-\frac{\gamma_2-\beta}{\alpha_2-x}+2B_2Y_2\right)dY_2\\
&+\left(Y_2-\frac{\alpha_2-\gamma_2}{x-\beta}-\frac{\alpha_2-x}{\gamma_2-\beta}W_2+2C_2Z\right)dZ,\\
dE_{2367}=&\left(Y_2-\frac{\alpha_2-\gamma_2}{\alpha_1-\gamma_1}Y_1-\frac{\alpha_2-\alpha_1}{\gamma_2-\gamma_1}W_2+2CW_1\right)dW_1\\
&+\left(W_2-\frac{\alpha_2-\gamma_2}{\alpha_1-\gamma_1}W_1-\frac{\gamma_2-\gamma_1}{\alpha_2-\alpha_1}Y_2+2AY_1\right)dY_1\\
&+\left(Y_1-\frac{\alpha_1-\gamma_1}{\alpha_2-\gamma_2}Y_2-\frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2}W_1+2DW_2\right)dW_2\\
&+\left(W_1-\frac{\alpha_1-\gamma_1}{\alpha_2-\gamma_2}W_2-\frac{\gamma_1-\gamma_2}{\alpha_1-\alpha_2}Y_1+2BY_2\right)dY_2.
\end{aligned}$$

In the general case, we take the following inhomogeneous coordinates ;

$$Y_p=e_{2\,p}/e_{2\,k+2}\quad(k+2<p\leqq n),\qquad Z_i=e_{i\,k+2}/e_{2\,k+2}\quad(2<i\leqq k),$$
$$W_{ip}=e_{i\,p}/e_{2\,k+2}\quad(2<i\leqq k,\ k+2<p\leqq n).$$

Then, similarly as in the case of $(k,n)=(3,7)$, from the quadratic equation $E_{2ik+2p}=0$, we can solve W_{ip} $(2<i\leqq k,\ k+2<p\leqq n)$ in terms of Y_p and Z_i on the Zariski open subset of $X(k,n)$. Substituting these into $E_{ijpq}=0$, we get non-trivial equations for Y's and Z's. Thus, at a generic point $x\in X(k,n)$, we obtain

$$\dim V(S_2(x))<n-4=\dim V(S_2),$$

which completes the proof of Theorem.

4. Disproof of a dream on $E(4,8;\{1/2\})$

The authors are afraid that the reader would not be satisfied by the argument in the previous section based on [S] and [Y], which are hardly elementary. So, in this section we give an elementary proof for $E(4,8;\{1/2\})$ that $Im(\varphi)$ does not lie in $Gr_{3,6} \subset \mathbb{P}^{19}$.

The idea is as follows: assume the contrary, then the *restriction* of the projective solution to any stratum consisting of degenerate 8-plane arrangements in P^3 has its image in quadratic hypersurfaces in a projective space, since Grassmannians can be defined only by quadratic equations. If we choose a 1-dimensional stratum, the restricted equation is an ordinary differential equation; so we can know whether its image lies in a quadric by the vanishing of the Laguerre-Forsyth invariant.

Let us carry out the above program. We consider the degenerate stratum given by the following matrix:

$$\begin{pmatrix} 1 & & & & 1 & & & 1 \\ & 1 & & & -1 & 1 & & \\ & & 1 & & & -1 & 1 & \\ & & & 1 & & & -1 & -x \end{pmatrix},$$

where each column defines a hyperplane. The integral belonging to the stratum is of the form

$$\int t_1^{\alpha_1-1}t_2^{\alpha_2-1}t_3^{\alpha_3-1}(1-t_1)^{\alpha_4-1}(t_1-t_2)^{\alpha_5-1}(t_2-t_3)^{\alpha_6-1}(1-xt_3)^{\alpha_7-1}dt_1\wedge dt_2\wedge dt_3.$$

The associated ordinary differential equation in x is of fourth order and coincides with the so-called generalized hypergeometric differential equation ${}_4E_3(a_1,a_2,a_3,a_4;b_1,b_2,b_3)$:

$$\theta(\theta+b_1-1)(\theta+b_2-1)(\theta+b_3-1)z - x(\theta+a_1)(\theta+a_2)(\theta+a_3)(\theta+a_4)z = 0,$$

where $\theta = xd/dx$ (refer to [E]), which admits the solution given by the following power series:

$${}_4F_3(a_1,a_2,a_3,a_4;b_1,b_2,b_3;x) = \sum_{n=0}^{\infty}\frac{(a_1,n)(a_2,n)(a_3,n)(a_4,n)}{(b_1,n)(b_2,n)(b_3,n)(1,n)}x^n,$$

where

$$a_1 = \alpha_1+\alpha_2+\alpha_3+\alpha_5+\alpha_6-2,\quad a_2 = \alpha_2+\alpha_3+\alpha_6-1,\quad a_3=\alpha_3,\quad a_4 = 1-\alpha_7,$$
$$b_1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6-2,\quad b_2 = \alpha_2+\alpha_3+\alpha_5+\alpha_6-1,\quad b_3 = \alpha_3+\alpha_6,$$

and $(a,n) = a(a+1)\cdots(a+n-1)$.

Now, consider the case where all α_i are equal to $1/2$; the corresponding parameters are $a_1=a_2=a_3=a_4=1/2$ and $b_1=b_2=b_3=1$. The question is to see if the curve in $\mathbb{P}^3$ defined by the ${}_4E_3$ lies on quadratic surfaces for this special choice of parameters. To proceed further, we need to recall a bit of the Laguerre-Forsyth theory. We start with an ordinary differential equation of the form

$$y^{\cdot\cdot\cdot\cdot} + 4p_1y^{\cdot\cdot\cdot} + 6p_2y^{\cdot\cdot} + 4p_3y^{\cdot} + p_4y = 0,$$

where y is the indeterminate of the variable x and the dot denotes the derivation relative to x. We can find a non-vanishing function λ and a new variable t so that the function $z = \lambda y$ relative to the coordinate t satisfies the ordinary differential equation

$$z'''' + 4r_3 z' + r_4 z = 0, \tag{4.1}$$

where r_3 and r_4 are differential polynomials of p_i, and $'$ denotes the derivation relative to t. The Laguerre-Forsyth theory (refer to, say, [MSY2], [W]) tells us that

$$\theta_3 = r_3 dt^3 \qquad \text{and} \qquad \theta_4 = (r_4 - 2r_3')dt^4$$

are projective invariants; that is, independent of the choice of such a coordinate t. For the case ${}_4E_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1)$, a calculation shows $r_3 = 0$.

On the other hand, for the ordinary differential equation

$$z'''' + rz = 0,$$

we can check that

$$I = \frac{(8rr'' - 9(r')^2)^2}{r^5}$$

is an absolute invariant; in our case it is equal to

$$I = -\frac{16(125x^6 - 4650x^5 + 3075x^4 - 38572x^3 + 3075x^2 - 4650x + 125)^2}{x(5x+1)^5(x+5)^5}.$$

In particular, I is not constant.

We next consider the case where the projective curve defined by the equation (4.1) is on a nondegenerate quadratic surface, say, $\zeta_1\zeta_4 = \zeta_2\zeta_3$ in $\mathbb{P}^3(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$. Then around a generic point, we can choose a coordinate t so that the set of independent solutions is $\{1, t, f, tf\}$ for a function f. This means that the equation (4.1) is the tensor product of two differential equations

$$y_1'' = 0 \qquad \text{and} \qquad y_2'' = \frac{f''}{f'} y_2';$$

namely, $y_1 y_2$ are general solutions of (4.1). Such an ordinary differential equation is studied by [Ha] and its general form is known to be

$$z'''' - 2gz''' - 2g'z' + (g^2 - g'' - c^2)z = 0,$$

where g is a function and c is a constant. The invariants r_3 and r_4 of this equation are given by

$$r_3 = \frac{1}{2}g', \quad r_4 = 4c^2 - \frac{1}{5}g'' - \frac{36}{25}g^2.$$

If the image curve of a projective solution of the equation ${}_4E_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1)$ lies on a nondegenerate quadratic surface, since $r_3 = 0$, the function g must be constant, and so r_4 should also be constant, which implies $I = 0$. Therefore, our curve does not lie on any nondegenerate quadratic surface.

Suppose that the image $Im\varphi$ is on the Grassmannian $Gr_{3,6}$, then the image of a projective solution of the restricted system ${}_4E_3$ would be in the intersection $Gr_{3,6} \cap L$ of $Gr_{3,6}$ and a 3-dimensional linear subvariety L of $\mathbb{P}^{20-1}$. Since Grassmannians can be defined only by quadrics, the curve $Gr_{3,6} \cap L$ in L must be the intersection of two quadric surfaces. If the pencil generated by two quadric surfaces consists of degenerate quadrics only, the intersection must be linear, which contradicts that the projective solution is defined by linearly independent solutions.

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