MONODROMY OF THE HYPERGEOMETRIC DIFFERENTIAL EQUATION OF TYPE $(3,6)$ III

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Abstract. For the hypergeometric system $E(3,6;\alpha)$ of type $(3,6)$, two special cases $\alpha \equiv \mathbf{1/2}$ and $\alpha \equiv \mathbf{1/6}$ are studied in [MSY] and [MSTY2], respectively. The monodromy group of the former is an arithmetic group acting on a symmetric domain, and that of the latter is the unitary reflection group ST34. In this paper, we find a relation between these two groups.

Contents

1. Introduction

For the hypergeometric system $E(3,6;\alpha)$ of type $(3,6)$, two special cases $\alpha \equiv \mathbf{1/2}$ and $\alpha \equiv \mathbf{1/6}$ are studied in [MSY] and [MSTY2], respectively. The monodromy group of the former, say $M(1/2)$, is an arithmetic group acting on a symmetric domain, and that of the latter, say $M(1/6)$, is the unitary reflection group ST34. In this paper, we find a relation between these two groups; roughly speaking, $M(1/6)$ is isomorphic to $M(1/2)$ modulo 6.

2. Hypergeometric system $E(3,6;\alpha)$

Let $X = X(3,6)$ be the configuration space of six lines in the projective plane $\boldsymbol{P}^2$ defined as

$$X(3,6) = \mathrm{GL}_3(\mathbb{C})\backslash\{z \in M(3,6) \mid D_z(ijk) \neq 0, 1 \leq i < j < k \leq 6\}/H_6,$$

where $M(3,6)$ is the set of 3×6 complex matrices, $D_z(ijk)$ is the (i,j,k)-minor of z, and $H_6 \subset \mathrm{GL}_6(\mathbb{C})$ is the group of diagonal matrices. It is a 4-dimensional complex manifold.

A matrix $z \in M(3,6)$ defines six lines in $\boldsymbol{P}^2$:

$$L_j : \ell_j := z_{1j}t^1 + z_{2j}t^2 + z_{3j}t^3 = 0, \quad 1 \leq j \leq 6,$$

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where $t^1 : t^2 : t^3$ is a system of homogeneous coordinates. For parameters $\alpha = (\alpha_1, \dots, \alpha_6)$ ($\sum \alpha_i = 3$), we consider the integral

$$\int_\sigma \prod_{j=1}^{6} \ell_j(z)^{\alpha_j - 1} dt, \quad dt = t^1 dt^2 \wedge dt^3 + t^2 dt^3 \wedge dt^1 + t^3 dt^1 \wedge dt^2$$

for a (twisted) 2-cycle σ in $\boldsymbol{P}^2 - \cup L_j$. It is a function in z, but not quite a function on X. So for simplicity, we fix local coordinates on X as

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x^1 & x^2 \\ 0 & 0 & 1 & 1 & x^3 & x^4 \end{pmatrix},$$

and consider such integrals above as functions in $x = (x^1, x^2, x^3, x^4)$. Then they satisfy a system of linear differential equations on X, called **the hypergeometric differential equation** $E(3,6;\alpha)$ **of type** $(3,6)$. The rank (dimension of local solutions) of this system is six.

This system can be represented, for example, by

$$\begin{aligned}
&(\alpha_{234} - 1 + D_{1234})D_1 u = x^1(D_{13} + 1 - \alpha_5)(D_{12} + \alpha_2)u, \\
&(\alpha_{234} - 1 + D_{1234})D_2 u = x^2(D_{24} + 1 - \alpha_6)(D_{12} + \alpha_2)u, \\
&(\alpha_{234} - 1 + D_{1234})D_3 u = x^3(D_{13} + 1 - \alpha_5)(D_{34} + \alpha_3)u, \\
&(\alpha_{234} - 1 + D_{1234})D_4 u = x^4(D_{24} + 1 - \alpha_6)(D_{34} + \alpha_3)u, \\
&x^1(\alpha_5 - 1 - D_{13})D_2 u = x^2(\alpha_6 - 1 - D_{24})D_1 u, \\
&x^3(\alpha_5 - 1 - D_{13})D_4 u = x^4(\alpha_6 - 1 - D_{24})D_3 u, \\
&x^1(\alpha_2 + D_{12})D_3 u = x^3(\alpha_3 + D_{34})D_1 u, \\
&x^2(\alpha_2 + D_{12})D_4 u = x^4(\alpha_3 + D_{34})D_2 u, \\
&x^2 x^3 D_1 D_4 u = x^1 x^4 D_2 D_3 u,
\end{aligned}$$

where $D_i = x^i \partial/\partial x^i, \alpha_{i\dots j} = \alpha_i + \cdots + \alpha_j, D_{i\dots j} = D_i + \cdots + D_j$.

3. A compactification $\bar{X}$ of $X(3,6)$

The configuration space $X = X(3,6)$ admits an obvious action of the symmetric group S_6 permuting the numbering of the six lines.

The Grassmann duality on the Grassmannian $\mathrm{Gr}(3,6)$ induces an involution $*$ on X. A system of six lines, representing a point of X, is fixed by $*$ if and only if there is a conic tangent to the six lines. The set of fixed points of $*$ on X is a 3-dimensional submanifold of X. Indeed it is isomorphic to the configuration space $X(2,6)$ of six points on $\boldsymbol{P}^1$.

The action of S_6 and that of $*$ commutes. There is a compactification $\bar{X}$ of X on which $S_6 \times \langle * \rangle$ acts bi-regularly, such that $\bar{X}/\langle * \rangle \cong \boldsymbol{P}^4$ (bi-regular).

4. Local property of $E(3,6)$

Let X_{ijk} be the set of points in $\bar{X}$ represented by the system $(L_1, \dots, L_6)$ of six lines such that L_i, L_j, L_k meet at a point. Then

$$X = \bar{X} - \cup_{1 \le i < j < k \le 6} X_{ijk}.$$

The system $E(3,6)$ can be considered to be defined on $\bar{X}$ with regular singularity along X_{ijk}.

The **Schwarz map** $s(\alpha)$ of the system $E(3,6;\alpha)$ is defined by linearly independent solutions $u_1,\dots,u_6$ as

$$s : X \ni x \longmapsto u_1(x) : \cdots : u_6(x) \in \boldsymbol{P}^5,$$

It has exponent $\alpha_i+\alpha_j+\alpha_k-1$ along X_{ijk}.

In the x-coordinates, only 14 among the twenty X_{ijk} are visible; they are given by the equations $D(ijk)=0$:

$$\begin{array}{llll} D(135)=x^1, & D(136)=x^2, & D(345)=x^1-1, & D(346)=x^2-1,\\ D(125)=x^3, & D(126)=x^4, & D(245)=x^3-1, & D(246)=x^4-1,\\ D(145)=x^1-x^3, & D(146)=x^2-x^4, & D(256)=x^3-x^4, & D(356)=x^1-x^2, \end{array}$$

$$D(156)=x^1x^4-x^2x^3,\quad D(456)=(x^1-1)(x^4-1)-(x^2-1)(x^3-1).$$

5. Monodromy groups

For $1\le i<j<k\le 6$, we introduce 6-vectors a_{ijk} and b_{ijk}:

$$\begin{array}{ll}
a_{123} & =(-d_{123},d_{12}c_3,0,-d_1c_{23},0,0),\quad b_{123}=(1,0,0,0,0,0),\\
a_{124} & =(-d_4c_{12},-d_{12},d_{12}c_4,d_1c_2,-d_1c_{24},0),\quad b_{124}=(1,1,0,0,0,0),\\
a_{125} & =(-d_5c_{12},0,-d_{12},0,d_1c_2,0),\quad b_{125}=(1,1,1,0,0,0),\\
a_{126} & =(1,0,0,0,0,0),\quad b_{126}=(-d_{126},-d_{1236}/c_3,d_5c_{126},0,0,0),\\
a_{134} & =(d_4c_1,-d_{34}c_1,d_3c_{14},-d_1,d_1c_4,-d_1c_{34}),\quad b_{134}=(0,1,0,1,0,0),\\
a_{135} & =(d_5c_1,-d_5c_{13},-d_3c_1,0,-d_1,d_1c_3),\quad b_{135}=(0,1,1,1,1,0),\\
a_{136} & =(-1/c_2,c_3/c_2,0,0,0,0),\quad b_{136}=(-d_2,-d_{1236}/c_3,d_5c_{126},-d_{1236}/c_3,d_5c_{126},0),\\
a_{145} & =(0,d_5c_1,-d_{45}c_1,0,0,-d_1),\quad b_{145}=(0,0,1,0,1,1),\\
a_{146} & =(0,1,-c_4,0,0,0),\quad b_{146}=(d_2/c_2,-d_{1456},-d_5c_{16},d_3c_{1456},-d_5c_{16},-d_5c_{16}),\\
a_{156} & =(0,0,1,0,0,0),\quad b_{156}=(d_2/c_2,-d_{1456},-d_{156},d_3c_{1456},d_{34}c_{156},d_4c_{156}),\\
a_{234} & =(-d_4,d_{34},-d_3c_4,-d_{234},d_{23}c_4,-d_2c_{34}),\quad b_{234}=(0,0,0,1,0,0),\\
a_{235} & =(-d_5,d_5c_3,d_3,-d_5c_{23},-d_{23},d_2c_3),\quad b_{235}=(0,0,0,1,1,0),\\
a_{236} & =(1,-c_3,0,c_{23},0,0),\quad b_{236}=(d_1/c_1,0,0,d_{45}c_6,d_5c_6,0),\\
a_{245} & =(0,-d_5,d_{45},d_5c_2,-d_{45}c_2,-d_2),\quad b_{245}=(0,0,0,0,1,1),\\
a_{246} & =(0,1,-c_4,-c_2,c_{24},0),\quad b_{246}=(d_1/c_1,d_1/c_1,0,-d_3c_{456},d_5c_6,d_5c_6),\\
a_{256} & =(0,0,1,0,-c_2,0),\quad b_{256}=(d_1/c_1,d_1/c_1,d_1/c_1,-d_3c_{456},-d_{34}c_{56},-d_4c_{56}),\\
a_{345} & =(0,0,0,-d_5,d_{45},-d_{345}),\quad b_{345}=(0,0,0,0,0,1),\\
a_{346} & =(0,0,0,1,-c_4,c_{34}),\quad b_{346}=(0,d_1/c_1,0,-d_{3456},0,d_5c_6),\\
a_{356} & =(0,0,0,0,1,-c_3),\quad b_{356}=(0,d_1/c_1,d_1/c_1,-d_{3456},-d_{3456},-d_4c_{56}),\\
a_{456} & =(0,0,0,0,0,1),\quad b_{456}=(0,0,d_1/c_1,0,d_{12}/c_{12},-d_{456}),
\end{array}$$

where

$$c_j=\exp 2\pi i\alpha_j,\quad c_{ij\cdots}=c_ic_j\cdots,\quad d_{ij\cdots}=c_{ij\cdots}-1.$$

The circuit matrix around a loop in X going once around the divisor X_{ijk} is given by

$$R_{ijk}=I_6-{}^ta_{ijk}\cdot b_{ijk}.$$

These R_{ijk} $(1\le i<j<k\le 6)$ generate the monodromy group $M(\alpha)$ of the system $E(3,6;\alpha)$. The monodromy group keeps the form

$$H=d_6\times\begin{pmatrix}
d_1d_2d_{345} & d_1d_2d_{45} & d_1d_2d_5 & 0 & 0 & 0\\
c_3d_1d_2d_{45} & d_1d_{23}d_{45} & d_1d_{23}d_5 & d_1d_3d_{45} & d_1d_3d_5 & 0\\
c_{34}d_1d_2d_5 & c_4d_1d_{23}d_5 & d_1d_{234}d_5 & c_4d_1d_3d_5 & d_1d_{34}d_5 & d_1d_4d_5\\
0 & c_2d_1d_3d_{45} & c_2d_1d_3d_5 & d_{12}d_3d_{45} & d_{12}d_3d_5 & 0\\
0 & c_{24}d_1d_3d_5 & c_2d_1d_{34}d_5 & c_4d_{12}d_3d_5 & d_{12}d_{34}d_5 & d_{12}d_4d_5\\
0 & 0 & c_{23}d_1d_4d_5 & 0 & c_3d_{12}d_4d_5 & d_{123}d_4d_5
\end{pmatrix}$$

invariant:

$${}^t\check{R}HR=H,\quad R\in M(\alpha),$$

where $\check{}$ is the operator which changes c_j to $1/c_j$. The above facts are shown in [MSTY1, MSTY2]. Note that the lists in these papers contains errors for a_{136} and a_{145}, so we tabulated here the corrected vectors.

The scalars $a_{ijk} \cdot {}^t b_{lmn}$ have the following properties.

Lemma 1. (1) $a_{ijk} \cdot {}^t b_{ijk} = 1 - c_i c_j c_k$ $(1 \leq i < j < k \leq 6)$.
(2) *For the other entries, we omit explicit expressions: If* $a_{ijk} \cdot {}^t b_{lmn}$ *is not zero as a rational function of* $c_1, \dots, c_5$ $(c_6 = (c_1 \cdots c_5)^{-1})$, *we replace the value simply by* z. *Then the matrix* $AB := (a_{ijk} \cdot {}^t b_{lmn})$ *is given as*

$$\begin{pmatrix}
z & z & z & z & z & z & z & 0 & 0 & 0 & z & z & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
z & z & z & z & z & z & z & z & z & 0 & z & z & z & z & z & 0 & 0 & 0 & 0 & 0 \\
z & z & z & z & 0 & z & z & z & z & z & 0 & z & z & z & z & z & 0 & 0 & 0 & 0 \\
z & z & z & z & 0 & 0 & z & 0 & z & z & 0 & 0 & z & 0 & z & z & 0 & 0 & 0 & 0 \\
z & z & 0 & 0 & z & z & z & z & z & 0 & z & z & z & z & z & 0 & z & z & 0 & 0 \\
z & z & z & 0 & z & z & z & z & z & z & 0 & z & z & z & z & z & z & z & z & 0 \\
z & z & z & z & z & z & z & 0 & z & z & 0 & 0 & z & 0 & z & z & 0 & z & z & 0 \\
0 & z & z & 0 & z & z & 0 & z & z & z & 0 & 0 & 0 & z & z & z & z & z & z & z \\
0 & z & z & z & z & z & z & z & z & z & 0 & 0 & 0 & 0 & z & z & 0 & z & z & z \\
0 & 0 & z & z & 0 & z & z & z & z & z & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & z & z \\
z & z & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & z & z & z & z & z & 0 & z & z & 0 & 0 \\
z & z & z & 0 & z & z & 0 & 0 & 0 & 0 & z & z & z & z & z & z & z & z & z & 0 \\
z & z & z & z & z & z & z & 0 & 0 & 0 & z & z & z & 0 & z & z & 0 & z & z & 0 \\
0 & z & z & 0 & z & z & 0 & z & 0 & 0 & z & z & 0 & z & z & z & z & z & z & z \\
0 & z & z & z & z & z & z & z & z & 0 & z & z & z & z & z & z & 0 & z & z & z \\
0 & 0 & z & z & 0 & z & z & z & z & z & 0 & z & z & z & z & z & 0 & 0 & z & z \\
0 & 0 & 0 & 0 & z & z & 0 & z & 0 & 0 & z & z & 0 & z & 0 & 0 & z & z & z & z \\
0 & 0 & 0 & 0 & z & z & z & z & z & 0 & z & z & z & z & z & 0 & z & z & z & z \\
0 & 0 & 0 & 0 & 0 & z & z & z & z & z & 0 & z & z & z & z & z & z & z & z & z \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & z & z & z & 0 & 0 & 0 & z & z & z & z & z & z & z
\end{pmatrix}.$$

We are interested in the most symmetric cases: $c_1 = \cdots = c_6 =: c$. Since $c^6 = 1$, we have

$$c = 1, \quad -1, \quad \omega, \quad -\omega,$$

where ω is a primitive third root of unity. Excluding the trivial case $c = 1$ (for all i, j, k, we have $a_{ijk} = 0$ or $b_{ijk} = 0$, and so $R_{ijk} = I_6$), there are three cases. By the explicit expression of AB, we see

Lemma 2. *When* $c = -1$ *and* $-\omega$, *the entries of the matrix* AB *marked* z *are not equal to zero. When* $c = \omega$, *the diagonal elements of the matrix* AB *are zero, while the other entries marked* z *are not zero.*

The case $c = \omega$ is of special interest; this will be studied in [SY]. In this paper, we treat the two cases

$$(\alpha_1, \dots, \alpha_6) = \begin{cases} \textbf{Case 0} \ : \mathbf{1/2} = (1/2, \dots, 1/2) \quad \text{and} \\ \textbf{Case 1} \ : \mathbf{1/6} = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6+1, 1/6+1). \end{cases}$$

In these cases, the circuit matrix R_{ijk} around the divisor X_{ijk} is a reflection of order 2 with respect to the hermitian matrix H; the vector b_{ijk} can be expressed in terms of a_{ijk} and H as

$$b_{ijk} = 2\bar{a}_{ijk} H / (a_{ijk}, a_{ijk})_H,$$

and so each reflection is expressed by a row 6-vector $a = a_{ijk}$ as

$$R_{ijk} = I_6 - 2 {}^t a \bar{a} H / (a, a)_H,$$

where $(a,a')_H := \bar{a}H{}^t a'$.

From now on, everything related to the case 0 has 0 on ones top-right, and put nothing for the case 1. For example,

$$c^0_j = -1, \quad d^0_j = -2, \quad d^0_{ij} = 0, \dots,$$

while

$$c_j = -\omega, \quad d_j = \omega^2, \quad d_{ij} = \omega^2 - 1, \dots \quad (\omega^2 + \omega + 1 = 0),$$

and for the case 0, the hermitian matrix is H^0, the roots are a^0_{ijk}, and the reflections are R^0_{ijk}, while for the case 1, they are H, a_{ijk} and R_{ijk}. Lemma 2 implies

Fact 1. *For vectors a_{ijk} and a^0_{ijk}, we have*

$$a_{ijk} \perp_H a_{lmn} \quad \textit{if and only if} \quad a^0_{ijk} \perp_{H^0} a^0_{lmn}.$$

Fact 2. ([MSY]) *The reflections R^0_{ijk} generate the principal congruence subgroup $\Gamma(2)$ with respect to H^0.*

Fact 3. ([MSTY2]) *The reflections R_{ijk} generate the finite complex reflection group ST34 (Shephard-Todd registration number 34, order* $39191040 = 2^9 \cdot 3^7 \cdot 5 \cdot 7$).

These groups $\Gamma(2)$ and ST34 will be studied in the next section.

6. Groups related to $ST34$ and $\Gamma(2)$

6.1. **Arithmetic groups.** The invariant form H^0 is an integral symmetric matrix unimodularly equivalent to

$$U \oplus U \oplus (-I_2), \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where I_k denotes the unit matrix of degree k. The symmetric domain $\boldsymbol{H}$ is defined to be a component of

$$\{z \in \mathbb{C}^6 \mid {}^t z H^0 z = 0,\ {}^t \bar{z} H^0 z > 0\} \subset \boldsymbol{P}^5.$$

We set

$$\begin{aligned} O_{H^0}(\mathbb{Z}) &= \{g \in GL_6(\mathbb{Z}) \mid\ {}^t g H^0 g = H^0\}, \\ \Gamma &= \{g \in O_{H^0}(\mathbb{Z}) \mid g \text{ keeps each of two connected components}\}, \\ \Gamma(2) &= \{g \in \Gamma \mid g \equiv I_6 \bmod 2\}, \\ \Gamma(3) &= \{g \in \Gamma \mid g \equiv I_6 \bmod 3\}, \\ \Gamma(6) &= \Gamma(2) \cap \Gamma(3) = \{g \in \Gamma \mid g \equiv I_6 \bmod 6\}. \end{aligned}$$

These groups act properly discontinuously on $\boldsymbol{H}$. Note that

$$-I_6 \in \Gamma(2), \quad -I_6 \notin \Gamma(3), \Gamma(6).$$

It is shown in [MSY] that the group $\Gamma(2)$ is generated by the reflections

$$R^0_a = I_6 - 2a\,{}^t a H^0/(a,a)_{H^0}$$

with respect to the roots a of norm $N(a) := -(a,a)_{H^0} = 1$ and that Γ is generated by reflections with respect to the roots of norm 1 and 2.

Now we define the subgroup $\Gamma(1)$ of Γ generated by reflections with respect to the roots of norm 1 and the products of two reflections with respect to the roots of norm 2. Note that

$$[\Gamma, \Gamma(1)] = 2, \quad \Gamma(2) \subset \Gamma(1).$$

6.2. **Finite groups ([Atlas]).** The invariant hermitian form H is negative definite. It is shown in [MSTY2] that the twenty reflections R_{ijk} generates a unitary reflection group often called ST34. It is a reflection group in $GL_6(\mathbb{Z}[\omega])$ with structure

$$ST34 = 6.G.2,$$

where

- 6 stands for the cyclic group of order 6,
- G stands for the simple group $PSU_4(3)$,
- 2 stands for the cyclic group of order 2,
- 6 is a normal subgroup of $ST34$ with $ST34/(6) \simeq G.2$ and
- $6.G$ is a normal subgroup of $ST34$ with $ST34/(6.G) \simeq 2$.

Note that 6 corresponds to the group $\langle -\omega I_6 \rangle$ generated by the scalar matrix $-\omega I_6$, and 2 corresponds to det $= \pm 1$, i.e., $S(ST34)/\langle -\omega \rangle$ is isomorphic to the simple group G, where $S(ST34)$ denotes the subgroup of $ST34$ with $\det(g) = 1$.

We set

$$GO_6^-(3) = \{g \in GL_6(\mathbb{F}_3) \mid {}^t g H g = H\}.$$

It is known that there exist two kinds of non-degenerate quadratic forms on $(\mathbb{F}_3)^6$ with Witt defect 0 and 1. Our H gives the form with Witt defect 1. It is shown in [Atlas] that this group has the structure

$$GO_6^-(3) = 2.G.(2^2).$$

Note that the center of $GO_6^-(3)$ is $\{\pm I_6\}$ and (2^2) corresponds to the characters $\det(g)$ and $\#_2(g)$, where $\#_2(g)$ means the spinor norm which is the number of reflections with $N(v_j) = 2$ modulo 2 when g is expressed as a product of reflections $R^0_{v_j}$ with $N(v_j) = 1, 2$.

We set

$$G\Omega_6^-(3) = \{g \in GO_6^-(3) \mid \#_2(g) = 0\}.$$

Since $-I_6 \in \Gamma(2)$, we have $\#_2(-I_6) = 0$. Note that the kernel of the natural map

$$p : \Gamma \to GO_6^-(3)$$

is $\Gamma(3)$.

7. Relation between the two monodromy groups

Proposition 1. *The correspondence*

$$R_{ijk} \longmapsto R^0_{ijk}$$

induces a homomorphism of $ST34/Z$ onto $\Gamma(2)/N$, where Z is the group generated by ωI_6 (index 2 subgroup of the center $\langle cI_6 \rangle$ of ST34), and N is a normal subgroup of $\Gamma(2)$ included in $\Gamma(6)$.

Proof. We first show that we can choose a set of generators of ST34 as

$$\text{GenRef} := \{R_{346},\ R_{245},\ R_{124},\ R_{123},\ R_{126},\ R_{156}\}.$$

Set

$$a_1 = a_{346},\ a_2 = a_{245},\ a_3 = a_{124},\ a_4 = a_{123},\ a_5 = a_{126},\ a_6 = a_{156}$$

and

$$R_1 = R_{346},\ R_2 = R_{245},\ R_3 = R_{124},\ R_4 = R_{123},\ R_5 = R_{126},\ R_6 = R_{156}.$$

Note that the inner products of the six roots are given as

$$((a_i, a_j)_H)_{i,j=1,\dots,6} = \begin{pmatrix} 2 & c & 1 & 0 & 0 & 0 \\ \bar{c} & 2 & 1 & 0 & \bar{c} & 0 \\ 1 & 1 & 2 & \bar{c} & 0 & 0 \\ 0 & 0 & c & 2 & 0 & 0 \\ 0 & c & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The six reflections are related as

$$\begin{array}{ccccccccc} R_1 & - & R_2 & - & R_3 & - & R_5 & - & R_6 \\ & & & & \backslash & 3 & / & & \\ & & & & & R_4 & & & \end{array}$$

This diagram reads: If two reflections $R, R' \in \mathrm{GenRef}$ are joined by an edge, then $(RR')^3 = I$, otherwise they commute. The node with label 3 means the following:

$$\{(a_3, a_4)_H (a_4, a_5)_H (a_5, a_3)_H\}^2 = \bar{c}^2 \quad (= \text{ third root of unity}),$$

and

$$(R_3 R_4 R_5)^2 (R_3 R_5 R_4)^2 = I.$$

The structure theorem for ST34 established in [Shephard] asserts that the six reflections with the above relations form a set of generating reflections. Moreover it is shown that a generator of the center of ST34 is given as

$$(R_1 R_2 R_3 R_4 R_5 R_6)^7 = cI.$$

We next show the corresponding relations for the reflections

$$R_1^0 = R_{346}^0,\ R_2^0 = R_{245}^0,\ R_3^0 = R_{124}^0,\ R_4^0 = R_{123}^0,\ R_5^0 = R_{126}^0,\ R_6^0 = R_{156}^0$$

hold modulo 6:

- For $R_a, R_b \in \mathrm{GenRef}$, if $(R_a R_b)^2 = I$ then $(R_a^0 R_b^0)^2 = I$. (Note that we do not need modulo 6.)
- For $R_a, R_b \in \mathrm{GenRef}$, if $(R_a R_b)^3 = I$ then $(R_a^0 R_b^0)^3 \equiv I \mod 6$.
- For the node with label 3, using the same notational convention as above,
$$(R_3^0 R_4^0 R_5^0)^2 (R_3^0 R_5^0 R_4^0)^2 \equiv I \quad \mod 6.$$
- For the center, we have
$$(R_1^0 R_2^0 R_3^0 R_4^0 R_5^0 R_6^0)^7 \equiv -I \quad \mod 6.$$

All the above relations can be shown by *computation.* □

Theorem 1. (1) $N = \Gamma(6)$ *and*

$$ST34/\langle \omega I_6 \rangle \simeq \Gamma(2)/\Gamma(6) \simeq \langle \Gamma(2), \Gamma(3) \rangle / \Gamma(3).$$

(2) $\Gamma(1) = \langle \Gamma(2), \Gamma(3) \rangle$ *and*

$$\Gamma/\Gamma(3) \simeq GO_6^-(3), \quad \Gamma(1)/\Gamma(3) \simeq G\Omega_6^-(3).$$

Proof. (1) Orders of $ST34/\langle \omega I_6 \rangle$ and $G\Omega_6^-(3)$ are equal to $4 \times |G|$. Consider the following maps

$$ST34 \xrightarrow{\varphi} \Gamma(2)/N \xrightarrow{f_1} \Gamma(2)/\Gamma(6) \xrightarrow{f_2} \langle \Gamma(2), \Gamma(3) \rangle / \Gamma(3) \xrightarrow{f_3} G\Omega_6^-(3),$$

where f_1, f_2 are naturally defined and f_3 is given by the natural projection

$$p : \langle \Gamma(2), \Gamma(3) \rangle \to G\Omega_6^-(3).$$

Note that f_1 is surjective and its kernel is $\Gamma(6)/N$, f_2 is bijective, and that f_3 is injective.

Put

$$f = f_3 \circ f_2 \circ f_1,$$

and consider its kernel M. Since M is normal in $\Gamma(2)/N \simeq ST34/\langle\omega\rangle$, there are few possibilities. Since the image of f has enough many elements, M does not contain G. We can easily see that $-I_6$ is mapped to $-I_6$ by f. By comparing the orders of $\Gamma(2)/N$ and $G\Omega_6^-(3)$, we conclude $M = I_6$ and f is bijective. Thus we conclude that f_3 is surjective, f_1 is injective and $N = \Gamma(6)$.

(2) It is clear that

$$\langle\Gamma(2), \Gamma(3)\rangle \subset \Gamma(1).$$

By the definitions of $\Gamma(1)$ and $G\Omega_6^-(3)$, we can regard $\Gamma(1)/\Gamma(3)$ as a subgroup of $G\Omega_6^-(3)$ by the natural projection p. Since f_3 is surjective,

$$p(\langle\Gamma(2), \Gamma(3)\rangle) \simeq \langle\Gamma(2), \Gamma(3)\rangle/\Gamma(3) \simeq G\Omega_6^-(3) \supset p(\Gamma(1)).$$

Thus we have $\langle\Gamma(2), \Gamma(3)\rangle \simeq \Gamma(1)$. □

8. Concluding remarks

8.1. **Geometric interpretation.** Since the domain $\boldsymbol{H}$ is simply connected, the Schwarz map $s(\mathbf{1/2}) : X \to \boldsymbol{H}$ can be thought of the universal branched covering branching along X_{ijk} with index 2. The Schwarz map $s(\mathbf{1/6})$ also branches along X_{ijk} with index 2. Thus, if $M(\subset \boldsymbol{P}^5)$ denotes the image of this Schwarz map, the composed map

$$s(\mathbf{1/6}) \circ s(\mathbf{1/2})^{-1} : \boldsymbol{H} \longrightarrow M$$

is single-valued. Moreover, the theorem above implies that this map induces a morphism

$$\boldsymbol{H}/\Gamma(6) \longrightarrow M.$$

8.2. **An elliptic analogue.** Recall the original hypergeometric differential equation

$$E(a, b, c) : x(1-x)u'' + \{c - (a+b+1)x\}u' - abu = 0,$$

and the Schwarz map

$$s(a, b, c) : \mathbb{C} - \{0, 1\} \ni x \longmapsto u(x) : v(x) \in \boldsymbol{P}^1,$$

where u and v are linearly independent solutions of $E(a, b, c)$. It is classically well known that the projective monodromy group of $E(1/2, 1/2, 1)$ is conjugate to the elliptic modular group $\Gamma_1(2)$, where

$$\Gamma_1(k) = \{g \in \mathrm{SL}_2(\mathbb{Z}) \mid g \equiv I_2 \mod k\}/\text{center},$$

which is a free group, and acts properly discontinuously and freely on the upper half-plane

$$\boldsymbol{H}_1 = \{\tau \in \mathbb{C} \mid \Im\tau > 0\},$$

and the Schwarz map $s(1/2, 1/2, 1)$ gives the developing map of the universal covering $\boldsymbol{H}_1 \to \mathbb{C} - \{0, 1\}$ inducing the isomorphism

$$\mathbb{C} - \{0, 1\} \cong \boldsymbol{H}_1/\Gamma_1(2).$$

On the other hand, the projective monodromy group of $E(1/6, -1/6, 1/3)$ is the tetrahedral group. Note that we have isomorphisms

$$\Gamma_1(2)/\Gamma_1(6) \cong \Gamma_1(1)/\Gamma_1(3) \cong \text{ tetrahedral group}.$$

Thus our main theorem can be thought of a generalization of this famous fact. Furthermore, this is not only an analogue: if we restrict the equations $E(3, 6; 1/2)$ and $E(3, 6; 1/6)$ to the singular strata X_{ijk}, $X_{ijk} \cap X_{lmn}, \dots$, we will end up with a 1-dim stratum, on which the monodromy groups of the two restricted equations (to both of which the Clausen formula

$$ {}_3F_2(2a, a+b, 2b; a+b+1/2, 2a+2b; x) = F(a, b; a+b+1/2; x)^2 $$

for the hypergeometric functions is applicable) are related as the above elliptic cases.

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