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LOCAL MOVES AND GORDIAN COMPLEXES

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ABSTRACT

By the works of Gusarov [2] and Habiro [3], it is known that a local move called the C_n move is strongly related to Vassiliev invariants of order less than n. The coefficient of the z^n term in the Conway polynomial is known to be a Vassiliev invariant of order n. In this note, we will consider to what degree the relationship is strong with respect to Conway polynomial. Let K be a knot, and K^{C_n} the set of knots obtained from a knot K by a single C_n move. Let $\nabla\mathcal{K}$ be the set of the Conway polynomials $\{\nabla_K(z)\}_{K\in\mathcal{K}}$ for a set of knots $\mathcal{K}$. Our main result is the following: There exists a pair of knots K_1, K_2 such that $\nabla K_1 = \nabla K_2$, and $\nabla K_1^{C_n} \neq \nabla K_2^{C_n}$. In other words, the C_n Gordian complex is not homogeneous with respect to Conway polynomial.

Keywords: C_n move; Gordian complex; Conway polynomial.

Mathematics Subject Classification 2000: 57M25

1. Introduction

In 1937, Wendt [21] introduced an operation for knots and links. We usually call the operation an *unknotting operation* (or briefly, a *crossing-change*), which is defined to be a local move between two knot diagrams K_1 and K_2 which are identical except near one crossing-point as in Fig. 1. Furthermore, we consider its spatial realization as follows: For two knots k_1 and k_2 represented by K_1 and K_2, k_1 and k_2 are said to be transformed into each other by an unknotting operation. Hirasawa and Uchida [4] introduced the *Gordian complex* by the unknotting operation as follows: We consider a knot as a 0-simplex (or vertex). For a positive integer m, we consider a set of m knots, each pair of which can be transformed into each other by a single unknotting operation, as an m-simplex. We regard this set of knots as a simplicial complex, which is called the Gordian complex. It is easily checked that the Gordian

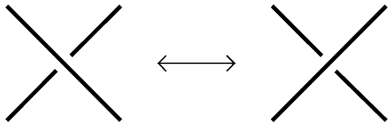

Fig. 1.

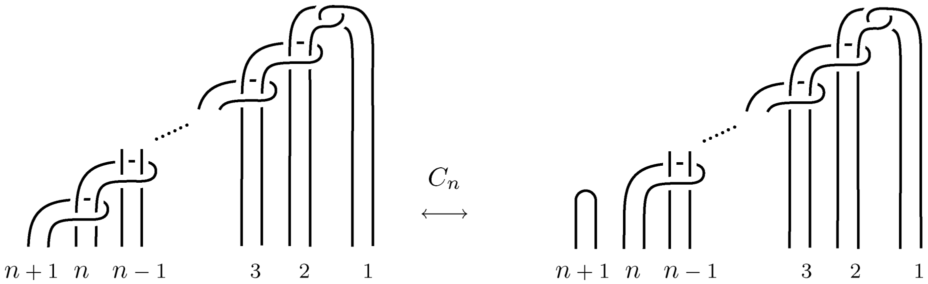

Fig. 2.

complex is connected. In [4] it is shown that every 1-simplex of the Gordian complex is a face of a simplex of arbitrary large dimension.

After Wendt, many local moves were introduced. In this note, we consider Habiro's C_n *moves* (cf. [3]), which are substitutions of $n + 1$ string trivial tangles as follows: For two link diagrams K and L which are identical except near one point as in Fig. 2, a local move between K and L is called a C_n move. Furthermore, we consider its spatial realization as follows: For two knots k_1 and k_2 represented by K_1 and K_2, k_1 and k_2 are said to be transformed into each other by a C_n move.

For $n = 1$, a C_n move is an unknotting operation. For $n = 2$, a C_n move is a Δ move defined by Matveev [8] and by Murakami and the first author [11]. For a local move λ, we can consider the λ Gordian complex in a parallel manner to that in [4]. We consider a knot as a 0-simplex (or vertex). For a positive integer m, we consider a set of m knots, each pair of which can be transformed into each other by a single local move λ, as an m-simplex. The λ *Gordian complex* might not be connected. The second author shows in [14] that every 0-simplex of the C_n Gordian complex is a face of a simplex of arbitrary large dimension.

For a knot invariant v which takes values in some abelian group, v can be extended to an invariant of singular knots, by the following: $v(K_D) = v(K_+) - v(K_-)$, where a singular knot is an immersion of a circle into $\mathbf{R}^3$ whose singularities are transversal double points. Here K_D, K_+, and K_- denote the diagrams of singular knots which are identical except near one crossing-point, as in Fig. 3. An invariant v is called a *Vassiliev invariant of order* n, and denote by v_n, if n is the smallest integer such that v vanishes on all singular knots with $n+1$ double points or more. If a knot invariant is a Vassiliev invariant of order m for some integer m,

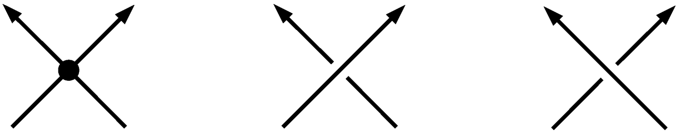

Fig. 3.

it is called an invariant of finite type. The Vassiliev invariants of order 0 and 1 are known to be trivial. The coefficient of the z^n term in the Conway polynomial is known to be a Vassiliev invariant of order n. By the works of Gusarov [2] and Habiro [3], it is known that a local move called the C_n move is strongly related to Vassiliev invariants of order less than n, as follows:

Proposition 1. *Two knots have the same value for each Vassiliev invariant of order less than n if and only if the two knots can be transformed into each other by a finite sequence of* C_n *moves.*

The following fact is an observation on a relationship between C_n moves and Conway polynomials.

Theorem 2. *There exists a pair of knots* K_1 *and* K_2 *such that* $\nabla K_1 = \nabla K_2$, *and that* $\nabla K_1^{C_n} \neq \nabla K_2^{C_n}$.

Here K^{C_n} means the set of knots obtained from a knot K by a single C_n move. $\nabla \mathcal{K}$ means the set of the Conway polynomials $\{\nabla_K(z)\}_{K \in \mathcal{K}}$ for a set of knots $\mathcal{K}$. The proof of Theorem 2 is given by a modification of the proofs of the following Theorems 2a, 2b, and 2c.

The alternative form for the C_2 move is given by the following Theorem 2a. The reason why the coefficients of the z^2 terms in the $\nabla_i(z)$'s are identical comes from the observation of Okada [12].

Theorem 2a. *For j polynomials with variables z,* $\nabla_i(z) = 1 + a_2 z^2 + a_4^{(i)} z^4 + \cdots + a_{2\ell_j}^{(i)} z^{2\ell_j}$ $(1 \leq i \leq j)$, *there exists a pair of knots* K_1 *and* K_2 *such that* $\nabla_{K_1}(z) = \nabla_{K_2}(z)$, $\nabla K_1^{C_2} \not\ni \nabla_1(z), \ldots, \nabla_j(z)$, *and* $\nabla K_2^{C_2} \ni \nabla_1(z), \ldots, \nabla_j(z)$.

For a generalization, we have the following result for the C_n moves for $n > 2$. The reason why the coefficients of the $z^2, \ldots, z^{2(n-1)}$ terms in the $\nabla_i(z)$'s are identical follows from a technical argument. Therefore there is a hope that the observation might be sharpened in the future.

Theorem 2b. *Let n be an integer larger than* 2. *For j polynomials with variables z,* $\nabla_i(z) = 1 + a_2 z^2 + \cdots + a_{2(n-1)} z^{2(n-1)} + a_{2n}^{(i)} z^{2n} + \cdots + a_{2\ell_j}^{(i)} z^{2\ell_j}$ $(1 \leq i \leq j)$, *there exists a pair of knots* K_1 *and* K_2 *such that* $\nabla_{K_1}(z) = \nabla_{K_2}(z)$, $\nabla K_1^{C_n} \not\ni \nabla_1(z), \ldots, \nabla_j(z)$, *and* $\nabla K_2^{C_n} \ni \nabla_1(z), \ldots, \nabla_j(z)$.

For the case $n = 1$ or for the unknotting operation, we have a rather weak observation as follows:

Theorem 2c. *For a polynomial with variables* z, $\nabla(z) = 1 + a_2 z^2 + \cdots + a_{2\ell} z^{2\ell}$, *there exists a pair of knots* K_1 *and* K_2 *such that* $\nabla_{K_1}(z) = \nabla_{K_2}(z)$, $\nabla K_1^{C_1} \not\ni \nabla(z)$, *and* $\nabla K_2^{C_1} \ni \nabla(z)$.

The above is just an observation, but it is a starting point for studing what kinds of obstruction might make such a situation.

Remark. By a parallel argument, we can give the following result. Let K be a knot, and K^{kC_n} the set of knots obtained from a knot K by applying C_n moves at most k-times. There exists a pair of knots K_1, K_2 such that $\nabla K_1 = \nabla K_2$, and $\nabla K_1^{kC_n} \neq \nabla K_2^{kC_n}$.

2. Surgical Description

It is well-known that any knot can be transformed to a trivial knot by crossing-changes at suitable crossing-points. Every crossing-change is obtained by a ± 1 surgery along a small trivial knot around the crossing-point with linking number 0. Levine [6] and Rolfsen [16, 17] introduced a *surgery description* of a knot and a *surgical view* of the Alexander matrix and Alexander polynomial as follows:

Proposition 3. *Let* K *be a knot, and* K_0 *a trivial knot. Then there exist* n *disjoint solid tori* $T_1, \ldots, T_n$ *in* $S^3 - K_0$ *and a homeomorphism* ϕ *from* $S^3 -{}^\circ T_1 \cup \cdots \cup {}^\circ T_n$ *to itself such that*

(1) $\phi(K_0) = K$,
(2) $T_1 \cup \cdots \cup T_n$ *is a trivial link,*
(3) $\mathrm{lk}(T_i, K_0) = \mathrm{lk}(T_i, K) = 0$ *for each* i, *and*
(3) $\phi(\partial T_i) = \partial T_i$ *and* $\mathrm{lk}(\mu_i', T_i) = 1$, *where* $\mu_i \subset \partial T_i$ *is a meridian of* T_i *and* $\mu_i' = \phi^{-1}(\mu_i)$.

From a surgery description, we have a surgical view of the Alexander matrix of the knot as follows:

Proposition 4. *Let* K *be a knot. Then* K *has an Alexander matrix* $M_K = (m_{ij}(t))$ *of the following form*:

(1) $m_{ij}(t) = m_{ji}(t^{-1})$, *and* (2) $|m_{ij}(1)| = \delta_{ij}$,
where $\delta_{ij} = 1$ (if $i = j$,) 0 (if $i \neq j$) *is the Kronecker's delta.*

Here the size of M_K is given by the number n in Proposition 3. The Alexander polynomial of a knot K is given by the determinant of an Alexander matrix of K, up to units.

3. Proof of Theorem 2a

It is known that there is a close relationship between the Alexander polynomial $\Delta_K(t)$ and the Conway polynomial $\nabla_K(z)$ for a knot K: $\Delta_K(t) = \nabla_K(t^{-1/2} - t^{1/2})$. Let $\Delta_i(t) = \nabla_i(t^{-1/2} - t^{1/2}) \quad (1 \leq i \leq j)$. It is also known that any Alexander polynomial can be realized by a knot with unknotting number 1, as shown by Kondo [5] and Sakai [19]. For the polynomial $\nabla_{j+1}(z) = 1 - ja_2z^2$, let $\Delta_{j+1}(t) = \nabla_{j+1}(t^{-1/2} - t^{1/2})$. Let K^* be a knot with unknotting number 1 and $\Delta_{K^*}(t) = \prod_{i=1}^{j+1} \Delta_i(z)^2$. For the polynomial $\nabla_{j+2}(z) = 1 - (a_2 \pm 1)z^2$, let $\Delta_{j+2}(t) = \nabla_{j+2}(t^{-1/2} - t^{1/2})$. Let K^{**} be a knot with unknotting number 1 and $\Delta_{K^{**}}(t) = \Delta_{j+2}(t)$. Let $K_1 = K^*\#K^*\#K^*\#K^*\#K^{**}$. Then K_1, from a surgical viewpoint, has an Alexander matrix of the following form:

$$\begin{pmatrix} \prod_{i=1}^{j+1} \Delta_i(z)^2 & 0 & 0 & 0 & 0 \\ 0 & \prod_{i=1}^{j+1} \Delta_i(z)^2 & 0 & 0 & 0 \\ 0 & 0 & \prod_{i=1}^{j+1} \Delta_i(z)^2 & 0 & 0 \\ 0 & 0 & 0 & \prod_{i=1}^{j+1} \Delta_i(z)^2 & 0 \\ 0 & 0 & 0 & 0 & \Delta_{j+2}(z) \end{pmatrix}.$$

A C_2 move is realized by two crossing-changes (cf. [11]). If K_1' is obtained from K_1 by a single C_2 move, then K_1' is obtained from K_1 by two crossing-changes. Therefore K_1', from a surgical viewpoint, has an Alexander matrix of the following form:

$$\begin{pmatrix} \prod_{i=1}^{j+1} \Delta_i(z)^2 & 0 & 0 & 0 & 0 & * & * \\ 0 & \prod_{i=1}^{j+1} \Delta_i(z)^2 & 0 & 0 & 0 & * & * \\ 0 & 0 & \prod_{i=1}^{j+1} \Delta_i(z)^2 & 0 & 0 & * & * \\ 0 & 0 & 0 & \prod_{i=1}^{j+1} \Delta_i(z)^2 & 0 & * & * \\ 0 & 0 & 0 & 0 & \Delta_{j+2}(z) & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix}.$$

If $\Delta_{K_1'}(t) = \Delta_i(t)$, then we have that the determinant of the above matrix is $\pm\Delta_i(t)$. In the case $\Delta_i(t) \neq 1$, we consider the equation modulo $\Delta_i(t)^2$, which becomes a contradiction. In the case $\Delta_i(t) = 1$, we take another nontrivial $\Delta_{i'}(t)$ and consider the equation modulo $\Delta_{i'}(t)^2$, which also becomes a contradiction. Therefore we have $\nabla K_1^{C_2} \not\ni \nabla_1(z), \nabla_2(z), \ldots, \nabla_j(z)$.

Let K_2 be a knot with unknotting number 1 and $\Delta_{K_2}(t) = \Delta_{K_1}(t)$. By the following Lemma A, it can be seen that $\nabla K_2^{C_2} \ni \nabla_1(z), \nabla_2(z), \ldots, \nabla_j(z)$, completing the proof.

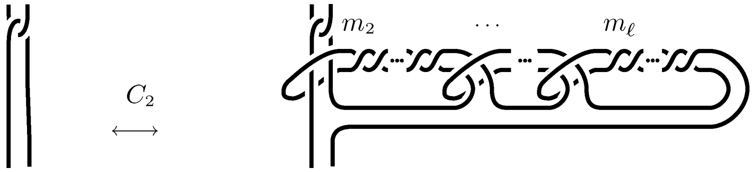

Fig. 4.

Lemma A. *Let K be a knot with algebraic unknotting number* 1*. For a set of integers $a'_2 = a_2(K) \pm 1$, and arbitrary integers a'_{2i} $(i = 2, 3, \ldots, \ell)$, there exists a knot $K' \in K^{C_2}$ with $\nabla_{K'}(z) = 1 + a'_2 z^2 + a'_4 z^4 + \cdots + a'_{2\ell} z^{2\ell}$.*

Here, a knot with *algebraic unknotting number* 1 means that a single crossing-change yields a knot with a trivial Alexander polynomial. This definition is different from the original one of Murakami [9], but Fogel [1] and Saeki [18] showed the equivalence of these definitions.

Proof. Since K is a knot with algebraic unknotting number 1, there exists a crossing at which the crossing-change yields a knot with a trivial Alexander polynomial. We consider such a crossing as in the left of Fig. 4. We transform this part of K to the right of Fig. 4 by a single C_2 move. Here, $m_2, \ldots, m_\ell$ are the numbers of left-handed full-twists. In the negative case $m_i < 0$, it means $|m_i|$ right-handed full-twists. By a parallel argument to that in Murakami [10], the difference of the Conway polynomials is $z^2 - (m_2+1)z^4 + \cdots + (-1)^{\ell-2}(m_{\ell-1}+1)z^{2\ell-2} + (-1)^{\ell-1}m_\ell z^{2\ell}$, completing the proof. □

We remark that the proofs of Lemmas A and B were inspired by a talk of Tsutsumi [20] at Tokyo Woman's Christian University and by the master's thesis of Makino [7].

4. Proof of Theorem 2b

The proof of Theorem 2b is quite similar to that of Theorem 2a. Here we assume that $n > 2$. Let $\Delta_i(t) = \nabla_i(t^{-1/2} - t^{1/2}) \quad (1 \le i \le j)$. Let K^* be a knot with unknotting number 1 and $\Delta_{K^*}(t) = \prod_{i=1}^{j} \Delta_i(z)^2$. Let $K_1 = K^* \# K^* \# K^*$. Then K_1, from a surgical viewpoint, has an Alexander matrix of the following form:

$$\begin{pmatrix} \prod_{i=1}^{j} \Delta_i(z)^2 & 0 & 0 \\ 0 & \prod_{i=1}^{j} \Delta_i(z)^2 & 0 \\ 0 & 0 & \prod_{i=1}^{j} \Delta_i(z)^2 \end{pmatrix}.$$

A C_n move is realized by two crossing-changes. If K'_1 is obtained from K_1 by a single C_n move, then K'_1 is obtained from K_1 by two crossing-changes. Therefore

K_1', from a surgical viewpoint, has an Alexander matrix of the following form:

$$\begin{pmatrix} \prod_{i=1}^{j} \Delta_i(z)^2 & 0 & 0 & * & * \\ 0 & \prod_{i=1}^{j} \Delta_i(z)^2 & 0 & * & * \\ 0 & 0 & \prod_{i=1}^{j} \Delta_i(z)^2 & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}.$$

If $\Delta_{K_1'}(t) = \Delta_i(t)$, then we have that the determinant of the above matrix is $\pm\Delta_i(t)$. In the case $\Delta_i(t) \neq 1$, we consider the equation modulo $\Delta_i(t)^2$, which becomes a contradiction. In the case $\Delta_i(t) = 1$, we take another nontrivial $\Delta_{i'}(t)$ and consider the equation modulo $\Delta_{i'}(t)^2$. Therefore we have $\nabla K_1^{C_n} \not\ni \nabla_1(z), \nabla_2(z), \ldots, \nabla_j(z)$.

Let K_2 be a knot with unknotting number 1 and $\Delta_{K_2}(t) = \Delta_{K_1}(t)$. By the following Lemma B, it can be seen that $\nabla K_2^{C_n} \ni \nabla_1(z), \nabla_2(z), \ldots, \nabla_j(z)$, completing the proof.

Lemma B. *Let K be a knot with algebraic unknotting number 1. For a set of arbitrary integers a'_{2i} ($i = n, \ldots, \ell$), there exists a knot $K' \in K^{C_n}$ with* $\nabla_K(z) - \nabla_{K'}(z) = \pm z^{2n-2} + a'_{2n}z^{2n} + \cdots + a'_{2\ell}z^{2\ell}$.

Proof. Since K is a knot with algebraic unknotting number 1, there exists a crossing at which the crossing-change yields a knot with a trivial Alexander polynomial. We consider such a crossing as in the top-left of Fig. 5. By an ambient isotopy,

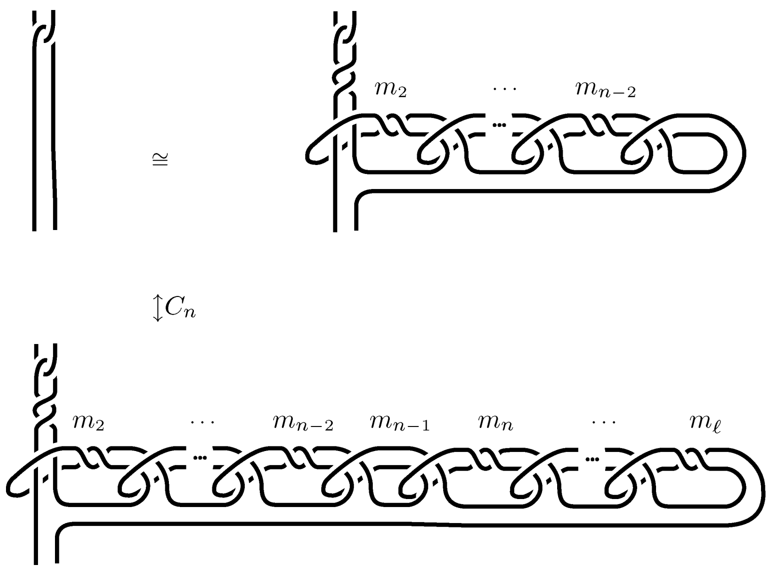

Fig. 5.

we deform this part of K to the top-right of Fig. 5. Then we transform this part of K to the bottom of Fig. 5 by a single C_n move. Here $m_2, \ldots, m_\ell$ are the numbers of left-handed full-twists. In the negative case $m_i < 0$, it means $|m_i|$ right-handed full-twists. Here $m_2, \ldots, m_{n-2}$ are assumed to be -1, and $m_{n-1} = 0$. By a parallel argument to that in [10], the difference of the Conway polynomials is $(-1)^{n-2}z^{2n-2} + (-1)^{n-1}(m_n+1)z^{2n} \cdots + (-1)^{\ell-2}(m_{\ell-1}+1)z^{2\ell-2} + (-1)^{\ell-1}m_\ell z^{2\ell}$, completing the proof. □

5. Proof of Theorem 2c

The proof of Theorem 2c is also quite similar to those of Theorems 2a and 2b.

First, we consider the case $\nabla(z) \neq 1$. Let $\Delta(t) = \nabla(t^{-1/2} - t^{1/2})$. Let K^* be a knot with unknotting number 1 and $\Delta_{K^*}(t) = \Delta(t)^2$. Let $K_1 = K^* \# K^*$. Then K_1, from a surgical viewpoint, has an Alexander matrix of the following form:

$$\begin{pmatrix} \Delta(t)^2 & 0 \\ 0 & \Delta(t)^2 \end{pmatrix}.$$

If K_1' is obtained from K_1 by a single crossing-change, then K_1', from a surgical viewpoint, has an Alexander matrix of the following form:

$$\begin{pmatrix} \Delta(t)^2 & 0 & * \\ 0 & \Delta(t)^2 & * \\ * & * & * \end{pmatrix}.$$

If $\Delta_{K_1'}(t) = \Delta(t)$, then we have that the determinant of the above matrix is $\pm\Delta(t)$. We consider the equation modulo $\Delta(t)^2$, which becomes a contradiction. Therefore we have $\nabla K_1^{C_1} \not\ni \nabla(z)$.

Let K^{**} be a knot with unknotting number 1 and $\Delta_{K^{**}}(t) = \Delta(t)$, and K^{***} a knot with unknotting number 1 and $\Delta_{K^{***}}(t) = \Delta(t)^3$. Let K_2 be the connected sum $K^{**} \# K^{***}$. It can be easily checked that $\nabla K_2^{C_1} \ni \nabla(z)$.

Next, we consider the case $\nabla(z) = 1$. We take a nontrivial Conway polynomial $\nabla'(z)$. Let $\Delta'(t) = \nabla'(t^{-1/2} - t^{1/2})$. Let K^* be a knot with unknotting number 1 and $\Delta_{K^*}(t) = \Delta'(t)^2$. Let $K_1 = K^* \# K^*$. Then K_1, from a surgical viewpoint, has an Alexander matrix of the following form:

$$\begin{pmatrix} \Delta'(z)^2 & 0 \\ 0 & \Delta'(z)^2 \end{pmatrix}.$$

If K_1' is obtained from K_1 by a single crossing-change, then K_1', from a surgical viewpoint, has an Alexander matrix of the following form:

$$\begin{pmatrix} \Delta'(z)^2 & 0 & * \\ 0 & \Delta'(z)^2 & * \\ * & * & * \end{pmatrix}.$$

If $\Delta_{K_1'}(t) = 1$, then we have that the determinant of the above matrix is ± 1. We consider the equation modulo $\Delta'(t)^2$, which becomes a contradiction. Therefore we have $\nabla K_1^{C_1} \not\ni 1$.

Let K_2 be a knot with unknotting number 1 and $\Delta_{K_2}(t) = \Delta'(t)^4$. It can be easily checked that $\nabla K_2^{C_1} \ni 1$, completing the proof.

Remark. In the preliminary note, we have discussed the following question: Let m_1, m_2 be sufficiently greater than n. Does there exist a pair of knots K_1, K_2 such that $\mathcal{V}_{m_1} K_1 = \mathcal{V}_{m_1} K_2$, and $\mathcal{V}_{m_2} K_1^{C_n} \neq \mathcal{V}_{m_2} K_2^{C_n}$. Here $\mathcal{V}_m$ means the set of Vassiliev invariants of order less than or equal to m $(m \geq 2)$, and $\mathcal{V}_m \mathcal{K}$ the value set $\{(v, \{v(K)\}_{K \in \mathcal{K}})\}_{v \in \mathcal{V}_{\Updownarrow}}$ for a set of knots $\mathcal{K}$. The question is still open.

Acknowledgments

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