Folding transformations for q-Painlevé equations

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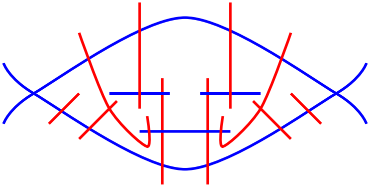
Joint work with Mikhail Bershtein, arXiv 2110.15320



Contents

q-Painlevé VI equation

q-difference dynamics on variables F, G depending on $a_0, \dots a_5$:

$$F\underline{F} = a_1^{-1}\frac{(G-a_3^{-1})(G-a_5^{-1}a_3^{-1})}{(G-1)(G-a_4)} \quad (1)$$

$$G\overline{G} = a_4\frac{(F-a_2)(F-a_0a_2)}{(F-1)(F-a_1^{-1})}. \quad (2)$$

Variables F, G could be viewed as functions on $a_0, \dots a_5$, such that

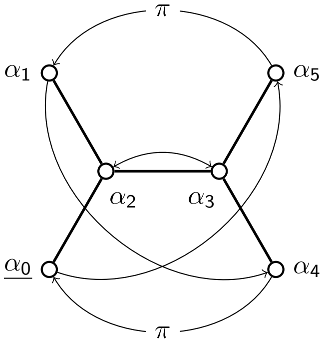
$$\begin{aligned} \overline{(F,G)}(a_0,\dots a_5) &= (F,G)(a_0,a_1,qa_2,q^{-1}a_3,a_4,a_5) \\ \underline{(F,G)}(a_0,\dots a_5) &= (F,G)(\dots q^{-1}a_2,qa_3,\dots) \end{aligned} \quad (3)$$

where $q=a_0a_1a_2^2a_3^2a_4a_5$. Actually, F, G depend on 4 parameters $a_{0,1,4,5}$ and independent variable $z=a_3^{-1}$, shifting on q.

Symmetries of q-Painlevé VI equation

- Symmetries are given by $W^e(D_5^{(1)})$ — affine Weyl group, extended by automorphisms.
- It acts on a_i as follows (thus called multiplicative root variables):

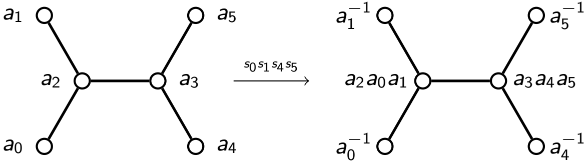
$$s_i(a_j) = a_j a_i^{-C_{ij}}, \quad C_{ij} \text{ is Cartan matrix}$$
$$\pi(a_i) = a_{\pi(i)}$$



$W^e(D_5^{(1)})$ also acts on functions F, G

	s_0	s_1	s_2	s_3	s_4	s_5	π
F	F	$a_1 F$	$a_2^{-1} F$	$F \frac{G-1}{G-a_3^{-1}}$	F	F	$1/G$
G	G	G	$G \frac{F-1}{F-a_2}$	$a_3 G$	$a_4^{-1} G$	G	F/a_2

Example: element $w = s_0 s_1 s_4 s_5$



$$F \mapsto a_1 F, \qquad G \mapsto a_4^{-1} G$$

- $w = s_0 s_1 s_4 s_5$ preserves q and commutes with the dynamics $a_2 \mapsto q a_2,\ a_3 \mapsto q^{-1} a_3, (F, G) \mapsto (\overline{F}, \overline{G})$.
- If $w(\vec{a}) = \vec{a}$ (e.g. $a_{0,1,4,5} = -1$), then the pairs (F, G) and $(w(F), w(G))$ (e.g. $(-F, -G)$) satisfy the same q-Painlevé VI.
- So, we obtain dynamics on w-invariant functions (e.g. gen. by F^2, FG, G^2) $\Rightarrow$ **some (another)** q-**Painlevé** (presumably).

Folding of q-Painlevé VI by $w = s_0 s_1 s_4 s_5$

$$
\begin{aligned}
F\underline{F} &= a_1^{-1}\frac{(G-a_3^{-1})(G-a_5^{-1}a_3^{-1})}{(G-1)(G-a_4)} \\
G\overline{G} &= a_4\frac{(F-a_2)(F-a_0a_2)}{(F-1)(F-a_1^{-1})}
\end{aligned}
$$

Taking $a_{0,1,4,5} = -1$, we obtain difference of squares in r.h.s.

Then introduce tilde half-shift $\widetilde{G} = F$, $\widetilde{\widetilde{G}} = \overline{G}$, $\widetilde{a_3} = q^{-1/2}a_3$

$$
\begin{aligned}
F\underline{F} &= \frac{G^2-a_3^{-2}}{G^2-1} \\
G\overline{G} &= \frac{F^2-a_2^2}{F^2-1}
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
\widetilde{G}\,\underset{\sim}{G} &= \frac{G^2-a_3^{-2}}{G^2-1} \\
G\widetilde{\widetilde{G}} &= \frac{\widetilde{G}^2-q^{1/2}a_3^{-2}}{\widetilde{G}^2-1}
\end{aligned}
$$

Parameterless q-Painlevé III equation

Introducing new variables $\mathbf{G} = G^2$, $\mathbf{F} = FG$

$$\mathbf{F}\underset{\sim}{\mathbf{F}} = \frac{\mathbf{G}(\mathbf{G} - a_3^{-2})}{\mathbf{G} - 1} \qquad \mathbf{G}\widetilde{\mathbf{G}} = \mathbf{F}^2 \tag{4}$$

we obtain q-Painlevé III with symmetry $A_1^{(1)}$ and corresponding variables

$$\mathbf{a}_0 = a_3^2, \quad \mathbf{a}_1 = a_2^2, \qquad \mathbf{q} = \mathbf{a}_0\mathbf{a}_1 = q^{1/2} \tag{5}$$

So we obtained **degree 2** algebraic transformation between two q-Painlevé

$$D_5^{(1)} \text{ eq.}|_{a_{0,1,4,5}=-1} \xrightarrow{/w=s_0s_1s_4s_5} A_1^{(1)} \text{ eq.} \tag{6}$$

Why foldings are interesting?

- For **differential** Painlevé foldings classified by Tsuda, Okamoto, Sakai '05. What about q-generalization?
- New **algebraic relations** between solutions of different types of (q-)Painlevé

$$(F(z), G(z)) \xrightarrow{\text{folding } \psi} (\mathbf{F}(z), \mathbf{G}(z)) = f(F(z), G(z)) \qquad (7)$$

  Way to connect special solutions: algebraic, Riccati, hypergeometric.
- Solutions of (q-)Painlevé written in terms of Nekrasov instanton partition functions $\mathcal{Z}$ (Gamayun, Iorgov, Lisovyi, '12; Jimbo, Nagoya, Sakai, '17; etc)

$$\mathcal{Z}_1 \to (F(z), G(z)), \quad \mathcal{Z}_2 \to (\mathbf{F}, \mathbf{G}) = f(F(z), G(z)) \quad \Rightarrow \mathcal{Z}_1 \approx \mathcal{Z}_2 \qquad (8)$$

  Way to relations on instanton partition functions of different theories.
- Answer to the q-Painlevé folding classification— engaging form by himself.

Contents

q-Painlevé equations and spaces of initial conditions

- Celebrated Sakai classification (Sakai '01):

$$q\text{-Painlevé eq.} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \text{blowed up in 8 points}$$

- These surfaces are known as **spaces of initial conditions**.
- Birational symmetries of surfaces and corresponding q-Painlevé equations — affine Weyl groups $W^e(E_{8-n}^{(1)})$, where $E_5 = D_5, E_4 = A_4 \dots$

$$\frac{A_0^{(1)}}{E_8^{(1)}} \to \frac{A_1^{(1)}}{E_7^{(1)}} \to \frac{A_2^{(1)}}{E_6^{(1)}} \to \frac{A_3^{(1)}}{D_5^{(1)}} \to \frac{A_4^{(1)}}{A_4^{(1)}} \to \frac{A_5^{(1)}}{(A_2+A_1)^{(1)}} \to \frac{A_6^{(1)}}{(2A_1)^{(1)}} \to \frac{A_7^{(1)}}{A_1^{(1)}}$$

- $A_{8-n}^{(1)} = (E_n^{(1)})^{\perp}$ in $E_8^{(1)}$.

Foldings: definition

Definition

Folding transformation *of q-Painlevé equation:*
Group element $w \in W^{ae}$ *and subset* $\mathcal{A}_w$ *of root variable space* $\mathcal{A} = (\mathbb{C}^*)^{r+1}$*:*

- $\mathcal{A}_w$ *is connected component of* w*-invariant subset:*

$$\mathcal{A}_w = (\vec{a} \in \mathcal{A} | w(\vec{a}) = \vec{a})$$

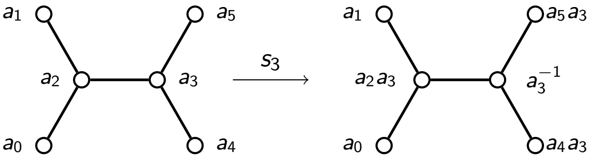
- *For generic* $\vec{a} \in \mathcal{A}_w$*:*

$$w((F,G)) \neq (F,G)$$

- *There exist translation (q-difference dynamics), that commutes with* w*:*

$$t \in P \subset W^{ae} : \ tw = wt.$$

Foldings: restrictions



$$s_3: \quad F \mapsto F\frac{G-1}{G-a_3^{-1}}, \qquad G \mapsto a_3 G$$

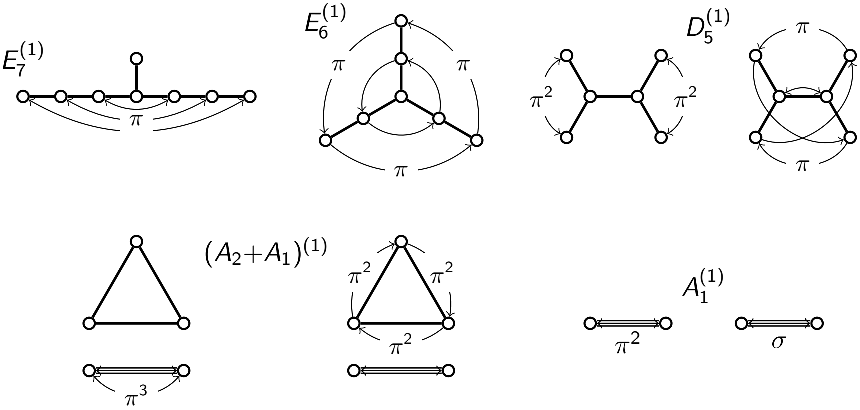
s_3 stabilizes $\vec{a}$, iff $a_3 = 1 \Rightarrow s_3$ acts trivially.

Lemma ($\sim$ Tsuda, Okamoto, Sakai)

For all symmetries $W(E_n^{(1)})$ of q-Painlevé equations

$$a_i = 1 \Rightarrow s_i(F, G) = (F, G)$$

But external automorphisms (group $\widehat{\Omega}$) of affine Weyl group give us foldings (both in differential and q-case). Here is the list of such folding group elements in q-case



Stabilizers of invariant subsets

Tsuda, Okamoto, Sakai '05: foldings in differential case come only from $\widehat{\Omega}$.
Differential Painlevé are parametrized by additive root variables x_i, $\sum x_i = 1$.

Lemma (Humphreys)

For $x_i \geq 0$ stabilizer of $\vec{x}$ is

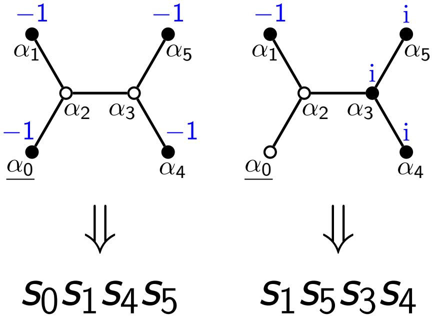
$$W^{ae}_{\vec{x}} \equiv (w \in W^{ae} | w(\vec{x}) = \vec{x}) = \widehat{\Omega}_{\vec{x}} \ltimes W^{\circ}_I, \tag{9}$$

where W°_i generated by reflections $s_i, i \in I$ and subset $I = (i \in \{0 \dots r\} | x_i = 0)$.

- It follows from lemma that product of reflections cannot give a folding in differential case.
- But, in q-difference case we obtain much more: we can build foldings as a **very special products of simple reflections**! (e.g. $w = s_0 s_1 s_4 s_5$).
- New type of foldings due to multiplicativity of root variables $\Rightarrow$ roots of unity.

Folding classification: colouring of Dynkin diagrams

New foldings: encoded by **Dynkin diagram colorings in black and white**



- Coloring denotes both the invariant subset $\vec{a}$ and group element w
- White points: arbitrary a_i.
- Black points (the subset I): $a_i =$ roots of unity.
- Subset I splits in connected (A-type) components.
- A_n connected component: Multiplier $s_1 s_2 \dots s_n$ in w.

Classificational theorem

For Weyl group S_{n+1} of A_n introduce group $\Omega \simeq C_{n+1}$:

$$\Omega = \langle s_1 s_2 \dots s_n \rangle = \langle (1 \dots n+1) \rangle \tag{10}$$

Theorem

Folding transformation w are of the form:

$$w \in \widehat{\Omega} \ltimes \prod \Omega_{\Phi_j} \tag{11}$$

where Φ_j are A-type connected components of black subset I for some coloring.

So, in general, folding is described not only by coloring but also by element in $\widehat{\Omega}$.

Folding classification: selection rules

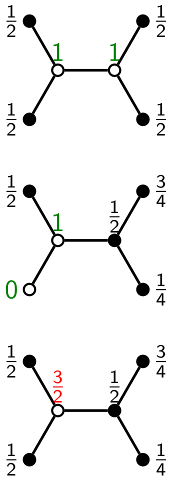
For $w \in \prod \Omega_{\Phi_j}$ we take colourings such that

- Black connected components are of A-type.
- Mark points of each black A_{n_j} by numbers $\phi_i = \frac{i m_j}{n_j+1}$, $i = 1 \dots n_j$ with some m_j, $\gcd(m_j, n_j + 1) = 1$.
- $\exists \{m_j\}$ such that for all white points p

$$\sum_{p' \in I \cap N(p)} \phi_{p'} \in \mathbb{Z}, \qquad (12)$$

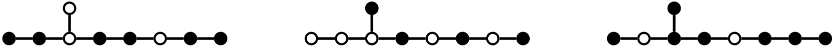
  where I is black subset, $N(p)$ is set of vertices, incident to (white) p.

On connected component Φ_j we have $a_i = e^{\frac{2\pi \mathrm{i} m_j}{n_j+1}}$.

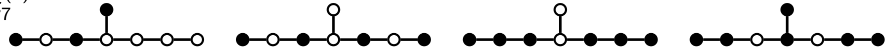


Folding classification: admissible colorings

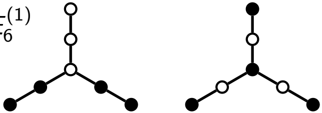
$E_8^{(1)}$



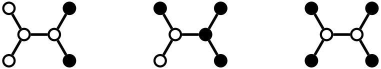
$E_7^{(1)}$

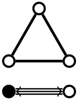


$E_6^{(1)}$

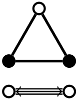


$D_5^{(1)}$



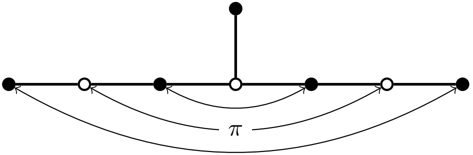


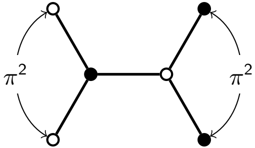
$(A_2+A_1)^{(1)}$



Folding classification: mixed case

Additionally we have 2 **mixed foldings** (1 for $E_7^{(1)}$ and 1 for $D_5^{(1)}$), containing both reflections and outer automorphisms. They are covered by certain generalization of the selection rule.





$w = \pi s_0 s_1 s_3$

$w = \pi^2 s_2 s_4$

Folding classification: "big" subgroups

Above foldings generate **cyclic folding subgroups**. In some cases we can construct bigger groups from the above foldings

Theorem

Only in above cases we have **non-cyclic folding subgroups**

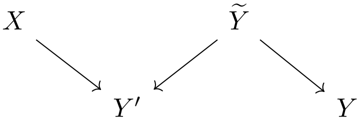
- *For $E_7^{(1)}$ we have two non-isomorphic folding subgroups $Dih_4 \simeq C_2 \ltimes C_4$*
- *For $D_5^{(1)}$ we have C_2^3.*

Any other non-cyclic subgroup is a subgroup of above.

Example: C_2^2 for $D_5^{(1)}$ could be obtained from $s_0 s_1 s_4 s_5$ and $s_4 s_5$, last generator acts as $(F, G) \mapsto (F, -G)$.

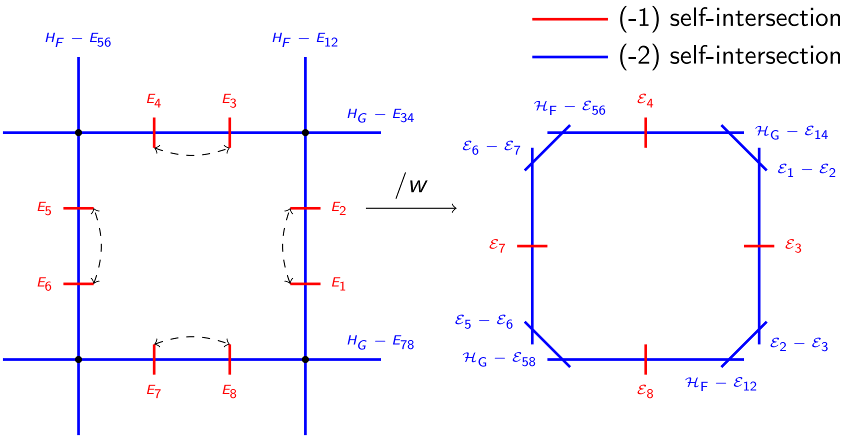
Contents

Now we want to find **Painlevé dynamics after taking quotient** over the folding group element, using spaces of initial conditions. The scheme is as follows:



- First we proceed to the quotient $Y' = X_{\vec{a}}/\langle w\rangle$.
- Usually it has toric singularities (corresponding to the fixed points of action).
- We resolve them in minimal way and obtain $\widetilde{Y}$.
- $\widetilde{Y}$ can be non-minimal, namely anticanonical class admits blowdown of (-1) components: $\widetilde{Y} \mapsto Y$.

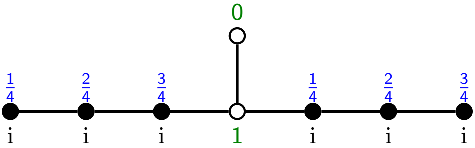
Simple example: $w = s_0 s_1 s_4 s_5$



- We start from $D_5^{(1)}/A_3^{(1)}$ surface and factor it over $(F,G) \mapsto (-F,-G)$
- 4 stationary points $(0,0),(0,\infty),(\infty,0),(\infty,\infty) \Rightarrow A_1$ singularities
- Resolving them, we recognize Sakai geometry of q-Painlevé $A_1^{(1)}/A_7^{(1)'}$

Another example: $w = s_{321}s_{765}$

Folding on $E_7^{(1)}/A_1^{(1)}$

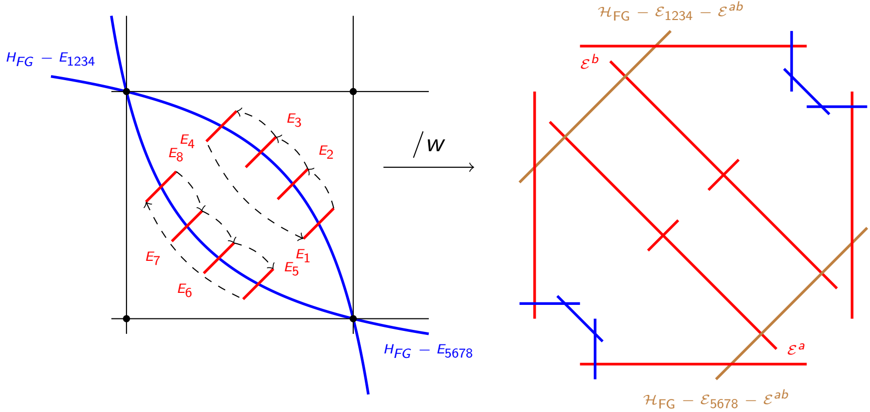


Action on coordinates is

$$F \mapsto -\mathrm{i}F, \qquad G \mapsto \mathrm{i}G. \tag{13}$$

Folding of order 4.

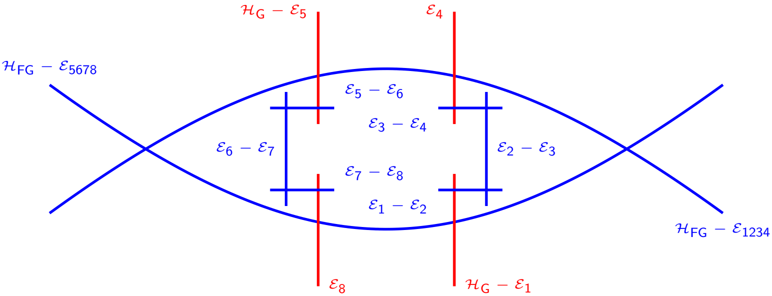
$w = s_{321}s_{765}$: quotient

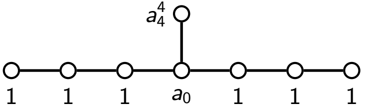


- Stationary points $(0,0), (\infty,\infty) \Rightarrow A_3$ singularities, $(0,\infty), (\infty,0) \Rightarrow (-4)$ singularities (brown divisor on the picture).
- We obtain non-minimal anticanonical class: contains (-1) curves $\mathcal{E}^a$ and $\mathcal{E}^b$.

$w = s_{321}s_{765}$: blow down

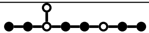
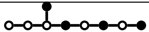
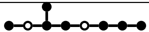
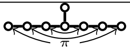
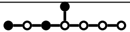
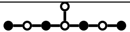
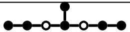
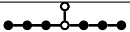
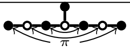
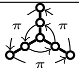
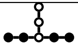
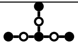
Blowing down $\mathcal{E}^a$ and $\mathcal{E}^b \Rightarrow E_7^{(1)}/A_1^{(1)}$ surface with parameters and coordinates



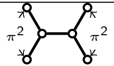
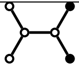
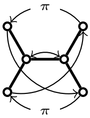
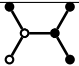
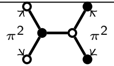
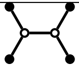


$$\mathbf{F} = \frac{FG - a_0}{FG - 1}\,\frac{G^4 - 1}{G^4 - a_4^{-4}},\qquad \mathbf{G} = \frac{FG - 1}{FG - a_0}. \tag{14}$$

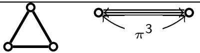
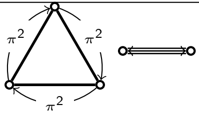
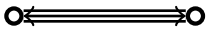
Final theorem I

Symm./surf.	Diagram	Ord.	Goes to	N. I.	Symmetry
$E_8^{(1)}/A_0^{(1)}$		3	$E_6^{(1)}/A_2^{(1)}$	$2A_2$	$C_2 \ltimes W^a_{A_2}$
		2	$E_7^{(1)}/A_1^{(1)}$	$3A_1$	$S_3 \ltimes W^a_{D_4}$
		4	$D_5^{(1)}/A_3^{(1)}$	A_3+A_1	$W^a_{A_1}$
$E_7^{(1)}/A_1^{(1)}$		2	$E_8^{(1)}/A_0^{(1)}$	D_4	$S_3 \ltimes (C_2^2 \times W^a_{D_4})$
				$4A_1$	$S_4 \ltimes W^a_{D_4}$
		2	$D_5^{(1)}/A_3^{(1)}$	$2A_1$	$C_2^2 \ltimes W^a_{A_3}$
		3	$E_3^{(1)}/A_5^{(1)}$	A_2	$W^{ae}_{A_1}$
		4	$E_7^{(1)}/A_1^{(1)}$	$2A_3$	$C_2 \times W^{ae}_{A_1}$
				D_6	$C_2^2 \times W^a_{A_1}$
$E_6^{(1)}/A_2^{(1)}$		3	$E_8^{(1)}/A_0^{(1)}$	E_6	$C_2 \ltimes (S_3 \times W^a_{A_2})$
				$3A_2$	$S_3 \ltimes W^a_{A_2}$
		2	$E_3^{(1)}/A_5^{(1)}$	A_1	$W^{ae}_{A_2}$

Final theorem II

Symm./surf.	Diagram	Ord.	Goes to	Nod. lat.	Symmetry
		2	$E_7^{(1)}/A_1^{(1)}$	D_4	$S_3 \ltimes (C_2^2 \times (C_2 \ltimes W^a_{3A_1}))$
				$4A_1$	$(S_3 \ltimes C_2^3) \ltimes W^a_{3A_1}$
$D_5^{(1)}/A_3^{(1)}$		4	$E_8^{(1)}/A_0^{(1)}$	E_7	$C_2^2 \times W^a_{A_1}$
				$2A_3+A_1$	$W^{ae}_{A_1}$
				D_6+A_1	$C_2^2 \times W^a_{A_1}$
		2	$A_1^{(1)}/A_7^{(1)'}$	$\varnothing$	$C_2 \times (C_4 \ltimes W^a_{A_1})$

Final theorem III

Symm./surf.	Diagram	Ord.	Goes to	N. I.	Symmetry
$E_3^{(1)}/A_5^{(1)}$		2	$E_6^{(1)}/A_2^{(1)}$	D_4	$C_2^2 \times W^{ae}_{A_2}$ (2δ)
				$4A_1$	$W^{ae}_{A_2}$
		3	$E_7^{(1)}/A_1^{(1)}$	E_6	$S_3 \times W^{ae}_{A_1}$, (3δ)
				$3A_2$	$W^{ae}_{A_1}$
$A_1^{(1)}/A_7^{(1)'}$	π^2 	2	$D_5^{(1)}/A_3^{(1)}$	D_4	$C_2^2 \times W^{a}_{A_1}$
	σ 			$4A_1$	$C_4 \ltimes W^{a}_{A_1}$

Remarks

- We find 4 of our 24 foldings in the literature: $E_6^{(1)} \leftrightarrows (A_2+A_1)^{(1)}$ and $E_7^{(1)} \leftrightarrows D_5^{(1)}$ (Ramani, Grammatikos, Tamizhmani '00)
- Natural question: to connect q-case foldings to the differential foldings. However, the $q \to 1$ limit is not straightforward: Folding $w = s_0 s_1 s_4 s_5 : qPVI \to qPIII_3$ goes to folding $PIII_1 \mapsto PIII_3$. Differential **limits depend on special configurations of $\vec{a}$**.
- q-Painlevé $\longleftrightarrow$ 5d SUSY gauge theories. Foldings $\Rightarrow$ covering of the **Seiberg-Witten curves** of the corr. theories.
- **Nodal curves** play an important role in the folding transformations.
- For the special configurations of $\vec{a}$ before and after folding we often have **projective reductions —** "fractional" translations (e.g. tilde for $s_0 s_1 s_4 s_5$).

Thank you for your attention!