Irregularity of Modified A-Hypergeometric Systems

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1 Introduction

We denote by $A = (a_{ij})$ a $d \times n$-matrix whose elements are integers and by $w = (w_1, \ldots, w_n)$ a vector of integers. We suppose that the set of the column vectors of A spans $\mathbb{Z}^d$. Set

$$\widetilde{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ & \cdots & & 0 \\ a_{dn} & \cdots & a_{dn} & 0 \\ w_1 & \cdots & w_n & 1 \end{pmatrix}.$$

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When we need to emphasize the dependency of $\widetilde{A}$ on w, we denote $\widetilde{A}$ by $\widetilde{A}(w)$. Throughout this paper, we do not always assume that A is pointed, but we assume that $\widetilde{A}$ is pointed. In other words, the column vectors of $\widetilde{A}$ lie in an open half-space with boundary passing through the origin in $\mathbb{R}^{d+1}$. Note that when A is pointed, then $\widetilde{A}$ is also pointed.

Definition 1 ([20]) We call the following system of differential equations $H_{A,w}(\beta)$ a *modified A-hypergeometric system*:

$$\left(\sum_{j=1}^{n} a_{ij}x_j\partial_j - \beta_i\right) \bullet f = 0, \qquad (i = 1, \ldots, d) \tag{1}$$

$$\left(\sum_{j=1}^{n} w_j x_j\partial_j - t\partial_t\right) \bullet f = 0, \tag{2}$$

$$\left(\prod_{i=1}^{n} \partial_i^{u_i} t^{u_{n+1}} - \prod_{j=1}^{n} \partial_j^{v_j} t^{v_{n+1}}\right) \bullet f = 0 \tag{3}$$

$$(\text{with } u, v \in \mathbb{N}^{n+1} \qquad \text{running over all } u, v \text{ such that } \widetilde{A}u = \widetilde{A}v).$$

Here, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{C}^d$ is a vector of parameters.

The modified system is defined on $\mathbb{C}^{n+1}$ with coordinates $(x_1, \ldots, x_n, t)$.

Let D be the Weyl algebra in $x_1, \ldots, x_n, t$. The left ideal in D generated by the operators in (1), (2), and (3) is also denoted by $H_{A,w}(\beta)$ if no confusion arises. The left D-module $D/H_{A,w}(\beta)$ is denoted by $M_{A,w}(\beta)$. The modified system is a variation of the A-hypergeometric system $H_A(\beta)$ (see, e.g., [8], [16], [20]).

Let us introduce one more modified system. Take a complex number α. Let $H_{A,w,\alpha}(\beta)$ be the left ideal of the Weyl algebra D generated by the operators in (1), (3) and $\sum_{j=1}^{n} w_j x_j \partial_j - t\partial_t - \alpha$. This modified A-hypergeometric system is also introduced in [20]. The D-module $M_{A,w,\alpha}(\beta) = D/H_{A,w,\alpha}(\beta)$ is, intuitively speaking, $t^{-\alpha}M_{A,w}(\beta)t^{\alpha}$. We know by [20] that the D-module $M_{A,w,\alpha}(\beta)$ is holonomic for any A, w, α and β.

A summand of the modified system agrees with the original A-hypergeometric system $H_A(\beta)$ on the space $t \neq 0$. Let us explain it. We denote $X = \mathbb{C}^n \times \mathbb{C} = \{(x, t)\}$ and $Y = \mathbb{C}^n \times \mathbb{C} = \{(y, s)\}$. Following [20, Section2] we consider the map

$$\varphi_w : Y^* := \mathbb{C}^n \times \mathbb{C}^* = \{(y, s)\} \longrightarrow X^* := \mathbb{C}^n \times \mathbb{C}^* \subset X \tag{4}$$

defined by $\varphi_w(y_1, \ldots, y_n, s) = (s^{-w_1}y_1, \ldots, s^{-w_n}y_n, s)$. We will denote simply $\varphi = \varphi_w$ if no confusion is possible.

The pullback image of the ideal $H_{A,w,\alpha}(\beta)$ by φ equals the ideal of differential operators on Y^* generated by $H_A(\beta)$ and $s\partial_s + \alpha$. Notice that this last ideal is nothing but the A-hypergeometric ideal associated with the matrix $\widetilde{A}(\mathbf{0})$ and the parameter vector $(\beta, -\alpha)$. Let us denote this ideal by $H_{\widetilde{A}(\mathbf{0})}(\beta, -\alpha)$. Notice also that this last ideal is independent on w.

Let $\mathcal{D}$ be the sheaf of holomorphic differential operators on X and $\mathcal{M}_{A,w,\alpha}(\beta)$ the quotient sheaf $\mathcal{D}/\mathcal{D}H_{A,w,\alpha}(\beta)$. By the previous observation, we have that the hypergeometric $\mathcal{D}$-module $\mathcal{M}_{\widetilde{A}(\mathbf{0})}(\beta, -\alpha) = \mathcal{D}/\mathcal{D}H_{\widetilde{A}(\mathbf{0})}(\beta, -\alpha)$ is the extension to Y of the pullback module

$\varphi^*(\mathcal{M}_{A,w,\alpha}(\beta))$ considered as a $\mathcal{D}_{Y^*}$–module on the first space $Y^* = \mathbb{C}^n \times \mathbb{C}^*$. Since φ is a biholomorphic map both $\mathcal{D}$-modules $\mathcal{M}_{\widetilde{A}(\mathbf{0})}(\beta, -\alpha)_{|Y^*}$ and $\mathcal{M}_{A,w,\alpha}(\beta)_{|X^*}$ are isomorphic.

Series solutions of the A-hypergeometric system $H_A(\beta)$ has been studied since Gel'fand, Kapranov, and Zelevinsky constructed convergent series solutions standing for the regular triangulation obtained by a generic weight vector w [8]. The construction is generalized as follows [16]: Assume that β is generic. Suppose that the initial ideal $\mathrm{in}_{(-w,w)}(H_A(\beta))$ has a solution of the form y^ρ, $\rho \in \mathbb{C}^n$. Then, the monomial y^ρ can be extended to a formal series solution $\phi(y) = y^\rho + \cdots$ of $H_A(\beta)$. We call the series $\phi(y)$ a series solution associated to the weight vector w. The series is divergent in general. We are interested in giving an explicit expression of a solution of $H_A(\beta)$ whose asymptotic expansion is $\phi(y)$. A standard method, in the theory of ordinary differential equations, to construct such expression is the Laplace integral representation and the Borel transformation of divergent series. This method has been successful in the study of global analytic properties of solutions of ordinary differential equations, see, e.g., the book by Balser [3] and its references. Then, it is a natural problem to construct a Laplace integral representation standing for the divergent series solution $\phi(y)$ associated to the weight vector w. Our modified system, which is a system of differential equations for $\phi(t^{w_1}x_1, \ldots, t^{w_n}x_n)$, will be used to give an answer to this problem and we will start to study the behavior of solutions of the modified system near $t = 0$.

A key ingredient of our study is that the modified A-hypergeometric system $H_{A,w,\alpha}(\beta)$ is transformed into the hypergeometric system $H_{\widetilde{A}(w)}(\beta, \alpha-1)$ associated with the matrix $\widetilde{A}(w)$ and parameter $(\beta, \alpha-1)$, by the formal inverse Fourier transformation $t \mapsto \partial_t$, $\partial_t \mapsto -t$, which enables us to study the modified system by the theory of A-hypergeometric systems (see Subsection 4.1). For example, consider the A-hypergeometric system defined for $A = (1, 2)$ and β. The original system $H_A(\beta)$ is generated by $x_1\partial_1 + 2x_2\partial_2 - \beta$ and $\partial_1^2 - \partial_2$. The modified system $H_{A,w}(\beta)$ for $w = (-1, -1)$ is generated by the three operators $x_1\partial_1 + 2x_2\partial_2 - \beta$, $-x_1\partial_1 - x_2\partial_2 - t\partial_t$, $\partial_1^2 t - \partial_2$, and the inverse Fourier transformation of $H_{A,w}(\beta)$ is generated by $x_1\partial_1 + 2x_2\partial_2 - \beta$, $-x_1\partial_1 - x_2\partial_2 + t\partial_t + 1$, $\partial_1^2\partial_t - \partial_2$ which is nothing but $H_{\widetilde{A}(w)}(\beta, -1)$.

We are interested in the behavior of solutions of $H_{A,w}(\beta)$ and $H_{A,w,\alpha}(\beta)$ near the hyperplane $t = 0$ in the space X and giving Laplace integral representations of solutions. The structure of the paper is the following:

In Section 2 we recall Y. Laurent's definition of the slopes of a finitely generated D-module with respect to a hypersurface.

In Section 3 we first recall the description of the slopes of an A-hypergeometric system by using the notion of umbrella introduced by Schulze and Walther in [17], then we summarize the construction of the Gevrey solutions given in [7] and extend some of these results to the case of the irregularity at infinity.

In Section 4 we prove that the formal inverse Fourier transform, with respect to T, of a modified hypergeometric system is a (classical) hypergeometric system and we use this fact to describe the slopes of the former by using the umbrella of the latter. We construct, associated with any slope of a modified system, a basis of its Gevrey solutions, modulo convergent power series, when the parameters β and α are very generic, see Theorem 6. Moreover, for $\beta \in \mathbb{C}^d$ very generic we construct a basis of formal series solutions of the modified system for any $\alpha \in \mathbb{C}$ and $w \in \mathbb{Z}^n$, see Theorem 5. Later, in Sections 5 and 6, we will prove that this solution is an asymptotic expansion of a Laplace integral representation of a solution by the Borel summation method (see, e.g., [3]) (Theorem 7 and Section 6). As an application, we give an asymptotic error evaluation of finite sums of formal series solutions of original A-hypergeometric systems with irregular singularities studied in, e.g., [7] and [8] (see Theorem 7, the inequality (17), and Section 6). The Example 4 illustrates the application for the simplest A.

We would like to point out that integral representations of holomorphic solutions of A-hypergeometric systems are given by A. Esterov and K. Takeuchi [6] by using the so called rapid decay homology cycles. Our Laplace integral representation is different from their representation: the integrand of our representation is an A-hypergeometric function associated to a homogenized configuration of A (Section 5).

In Section 6 we also prove that the study of the irregularity of the modified system along T allows us to give an analytic meaning to the Gevrey series solutions of the original hypergeometric system, along coordinate varieties, constructed in [7]. More precisely we prove (see Proposition 5) that they are asymptotic expansions of certain holomorphic solutions of hypergeometric system.

An interplay of algebra (slopes and formal series) and analysis (Borel summation method) is a main point of this paper. In order to make a comprehensible presentation to readers from several disciplines, we will often review some well-known facts to experts. We hope that our style is successful.

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2 Generalities on slopes

All the D–module appearing here will be left D-modules unless stated otherwise. We will denote by $T \subset \mathbb{C}^{n+1}$ the hyperplane defined by $t = 0$.

Let M be a finitely generated D-module. To the L-filtration on D we associate a *good* L-filtration on M, by means of a finite presentation. The associated $\mathrm{gr}^L(D)$-module $\mathrm{gr}^L(M)$ is then finitely generated. The radical of the annihilating ideal $Ann_{\mathrm{gr}^L(D)}(\mathrm{gr}^L(M))$, which is independent of the *good* L–filtration on M, defines an affine algebraic subset of the cotangent space $T^*\mathbb{C}^{n+1} = \mathbb{C}^{2n+2}$. This algebraic set is called the L-characteristic variety of M and it is denoted by $Ch^L(M)$. [1]

We are going to consider a special type of L-filtration: For any real number $r \in \mathbb{R}_{\geq 0}$ we denote by L_r either the linear form $L_r = F + rV$ or the filtration on D given by the $(2n+2)$–component weight vector $(0, \ldots, 0, -r, 1, \ldots, 1, 1+r)$ where $-r$ is placed in the $(n+1)^{th}$-component. Here F (resp. V) stands for the order filtration on D (resp. the Malgrange-Kashiwara filtration with respect to T) as in Section 4.1.

The F–filtration (resp. V–filtration) also induces a natural F–graduation (resp. V–graduation) on the L_r–graded ring $\mathrm{gr}^{L_r}(D)$ which is isomorphic to the polynomial ring $\mathbb{C}[x, \xi]$ in $(2n+2)$–variables.

It is known [12] that the irregularity of a holonomic system with respect to a smooth hypersurface is deeply related with the slopes of the system defined with respect to the given

[1] The results stated in this paragraph are well known in D-module theory and can be proven following [4] which treats the case of the order filtration (i.e. the F-filtration) in D. This has been done for example in [10] and [11, Section 3.2] in a slightly different situation.

hypersurface.

See [11] for the definition of L_r–characteristic variety and its main properties. Laurent's definition is done in the micro-characteristic setting for analytic D–modules. See also [2, Section 2] for an equivalent definition better adapted to effective computations for modules on the Weyl algebra.

Definition 2 [11, Section 3.4] *Let M be a finitely generated D–module and $T = \{t = 0\} \subseteq \mathbb{C}^{n+1}$. Consider the natural projection $\Pi : T^*\mathbb{C}^{n+1} \longrightarrow T$, i.e.*

$$\Pi(x_1, \ldots, x_n, t, \xi_1, \ldots, \xi_n, \xi_t) = (x_1, \ldots, x_n, 0)$$

For any real number $r > 0$, let $I_T^r(M)$ be the closure of the projection by Π of the irreducible components of the L_r–characteristic variety $Ch^{L_r}M \subset T^*\mathbb{C}^{n+1}$ that are not (F, V)–bihomogeneous. The real number $s = r + 1 > 1$ is said to be a slope of M along T at $p \in T$ if and only if $p \in I_T^r(M)$.

As proved by Y. Laurent, see [11, Section 3.4], any slope is a rational number and the set of slopes of M is finite. Moreover, Y. Laurent also proved *loc. cit.* that $s = r + 1$ is a slope of M along T at $p \in T$ if and only if, in a neighborhood of $\Pi^{-1}(p)$, $Ch^{L_{r'}}(M)$ is not locally constant for $r' \in (r - \epsilon, r + \epsilon)$ with $\epsilon > 0$ small enough.

3 Irregularity of A–hypergeometric systems

Recall that A is a $d \times n$ integer matrix of rank d whose columns $a_1, \ldots, a_n$ generate $\mathbb{Z}^d$ as $\mathbb{Z}$–module. We denote by $H_A(\beta)$ the hypergeometric ideal associated with A and the parameter vector $\beta \in \mathbb{C}^d$ [8] and by $M_A(\beta)$ the corresponding hypergeometric system (also known as GKZ–system). This system is the quotient of $D_n := \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$, the Weyl algebra of order n, by the left ideal $H_A(\beta)$. In this Section we denote $X = \mathbb{C}^n$.

3.1 Slopes of A–hypergeometric systems

We recall in this subsection some results in [17], where A is assumed to be pointed, i.e. the columns of A lie in a open half-space defined by a hyperplane passing through the origin in $\mathbb{R}^d$. These results will not be applied to our matrix A but only to the matrix $\widetilde{A}(w)$ (see Subsection 4.2), which we assume to be pointed throughout this article.

We denote by a_i the i-th column of A for $i = 1, \ldots, n$. The L-characteristic variety of the hypergeometric system $M_A(\beta)$ has been described, in a combinatorial way, by M. Schulze and U. Walther [17] for any pointed matrix A and any filtration $L = (u, v)$ such that $u_i + v_i = c > 0$ for all $i = 1, \ldots, n$.[2]

The main tool for their description is the the notion of (A, L)-umbrella, an abstract cell complex that we will define here for the sake of completeness. First of all, the (A, L)–umbrella only depends on A and on the coefficients v_i of the linear form $L = L_{(u,v)}$. We will assume first that $v_i > 0$ for all i.

Definition 3 [17, Def. 2.7] *The (A, L)–polyhedron Δ_A^L is the convex hull in $\mathbb{R}^d$ of the set $\{\mathbf{0}, a_1/v_1, \ldots, a_n/v_n\}$. The (A, L)-umbrella Φ_A^L is the set of faces of Δ_A^L which do not contain zero. In particular, Φ_A^L contains the empty face.*

By $\Phi_A^{L,k} \subset \Phi_A^L$ we denote the subset of faces of dimension k. We will identify each face σ of Φ_A^L with the set $\{i : a_i/v_i \in \sigma\}$ and with $\{a_i : a_i/v_i \in \sigma\}$. The (A, L)-umbrella is then an abstract cell complex.

When not all the v_i are strictly positive then both the definition of the (A, L)-polyhedron and the (A, L)-umbrella are a little bit more involved. We refer to [17, Def. 2.7] for these precise definitions in the general case. See also Subsection 4.2.

Theorem 1 [17, Cor. 4.17] *The L–characteristic variety of $M_A(\beta)$ is given by:*

$$Ch^L(M_A(\beta)) = \bigcup_{\tau \in \Phi_A^L} \overline{C_A^\tau} \tag{5}$$

where $\overline{C_A^\tau}$ is the closure of the conormal C_A^τ of the torus orbit $O_A^\tau = \{\xi \in T_0^*X = \mathbb{C}^n : \xi_i = 0 \text{ if } i \notin \tau,\ \xi_i = y^{a_i} \text{ if } i \in \tau, y \in (\mathbb{C}^*)^d\}$.

Moreover, it is proved in [17, Lemma 3.14] that $\overline{C_A^\tau}$ meets T_0^*X for all $\tau \in \Phi_A^L$. Thus Theorem 1 provides a description of the slopes of a hypergeometric system at the origin along any coordinate variety $Y \subset X$ via considering the 1–parameter family of filtrations $L_r = F + rV$, $r > 0$, where V is the V-filtration[3] along Y:

Corollary 1 [17, Cor. 4.18] *The real number $s = r + 1 > 1$ is a slope of $M_A(\beta)$ along Y at the origin if and only if $\Phi_A^{L_{r'}}$ is not locally constant at $r' = r$.*

Remark 1 *When $Y \subset X$ is a coordinate hyperplane and using the comparison results in [12] it is proved in [7] that the set of slopes $s = r+1$ of $M_A(\beta)$ along Y at the origin described in Corollary 1 coincide with the set of slopes of $M_A(\beta)$ along Y at any $p \in Y$.*

3.2 Gevrey solutions of A–hypergeometric systems at infinity.

In this subsection we first summarize how to construct the Gevrey solutions of $M_A(\beta)$ whose Gevrey index is equal to a given slope along the coordinate hyperplane $Y = \{x_n = 0\}$ [7]. Then, we will extend some of the results in [7] to the case of the irregularity of a hypergeometric system at infinity, i. e. along a coordinate hyperplane $x_n = \infty$. This construction will be used later in the study of the irregularity along T of a modified hypergeometric system.

Let us denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on $X = \mathbb{C}^n$. For $Y = \{x_n = 0\}$, we denote by $\mathcal{O}_{\widehat{X|Y}}$ the formal completion of $\mathcal{O}_X$ along Y, whose germs at $(p, 0) \in Y$ are of the form $f = \sum_{m \geq 0} f_m(x_1, \ldots, x_{n-1}) x_n^m$ where all the f_m are holomorphic functions in a

[2]The case $L = F$ was first done by A. Adolphson [1].

[3]The V–filtration with respect to the coordinate variety $Y = (x_1 = \cdots = x_\ell = 0)$ is defined by assigning the weight -1 (resp. the weight 1) to the variables x_i (resp. ∂_i) for $i = 1, \ldots, \ell$ and the weight 0 to the remainder variables.

common neighborhood of p. Notice that the restriction of $\mathcal{O}_X$ to Y, denoted by $\mathcal{O}_{X|Y}$, is a subsheaf of $\mathcal{O}_{\widehat{X|Y}}$. For any real number s, we can also consider the sheaf $\mathcal{O}_{\widehat{X|Y}}(s)$ of Gevrey series along Y of order s as the subsheaf of $\mathcal{O}_{\widehat{X|Y}}$ whose germs f at $(p,0) \in Y$ satisfy

$$\sum_{m\geq 0} \frac{f_m(x_1,\ldots,x_{n-1})}{m!^{s-1}} x_n^m \in \mathcal{O}_{X|Y,(p,0)}.$$

We will also consider the sheaf $\mathcal{O}_{\widehat{X|Y}}(<s)$ of Gevrey series along Y of order less than s. If f belongs to $\mathcal{O}_{\widehat{X|Y}}(s) \setminus \mathcal{O}_{\widehat{X|Y}}(<s)$ for some s, we say that the *index* of the Gevrey series f is s.

Let $A = (a_1 \;\cdots\; a_n)$ be a full rank matrix with $a_i \in \mathbb{Z}^d$ for all $i = 1,\ldots,n$ and $d \leq n$. Following [8] and [16], for any vector $v \in \mathbb{C}^n$ we can define a series

$$\phi_v = \phi_v(x) := \sum_{u\in N_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} x^{v+u} \tag{6}$$

where $v \in \mathbb{C}^n$ verifies $Av = \beta$ and $N_v = \{u \in \ker(A) \cap \mathbb{Z}^n :\; \operatorname{nsupp}(v+u) = \operatorname{nsupp}(v)\}$. Here $\ker(A) = \{u \in \mathbb{Q}^n :\; Au = 0\}$, $\operatorname{nsupp}(w) := \{i \in \{1,\ldots,n\} :\; w_i \in \mathbb{Z}_{<0}\}$ is the negative support of $w \in \mathbb{C}^n$, $[v]_u = \prod_i [v_i]_{u_i}$ and $[v_i]_{u_i} = \prod_{j=1}^{u_i}(v_i - j + 1)$ is the Pochhammer symbol for $v_i \in \mathbb{C}$, $u_i \in \mathbb{N}$.

The series ϕ_v is annihilated by the hypergeometric ideal $H_A(\beta)$ if and only if the negative support of v is minimal, i.e. $\nexists u \in \ker(A) \cap \mathbb{Z}^n$ with $\operatorname{nsupp}(v+u) \subsetneq \operatorname{nsupp}(v)$ (see [16, Section 3.4]).

When $\beta \in \mathbb{C}^d$ is very generic, i.e. when β is not in a countable union of Zarisky closed sets, there is a basis of the Gevrey solution space of $M_A(\beta)$ along Y given by series ϕ_v for suitable vectors $v \in \mathbb{C}^n$ (see Theorem 3).

For any subset $\eta \subseteq \{1,\ldots,n\}$ we denote by A_η the submatrix of A given by the columns of A indexed by η and we denote $\bar{\eta} = \{1,\ldots,n\} \setminus \eta$.

We say that $\sigma \subseteq \{1,\ldots,n\}$ is a d-simplex with respect to A (or simply that σ is a d-simplex) if the columns of A indexed by σ determine a basis of $\mathbb{R}^d$. We can reorder the variables in order to have $\sigma = \{1,\ldots,d\}$ without loss of generality. Then a basis of $\ker(A)$ associated with σ is given by the columns of the matrix:

$$B_\sigma = \begin{pmatrix} -A_\sigma^{-1} A_{\overline{\sigma}} \\ I_{n-d} \end{pmatrix} = \begin{pmatrix} -A_\sigma^{-1} a_{d+1} & -A_\sigma^{-1} a_{d+2} & \cdots & -A_\sigma^{-1} a_n \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}$$

A vector $v \in \mathbb{C}^n$ satisfying $v_i \in \mathbb{N}$ for all $i \notin \sigma$ and $Av = \beta$ can be written as

$$v^{\mathbf{k}} = (A_\sigma^{-1}(\beta - \sum_{i\notin\sigma} k_i a_i), \mathbf{k})$$

for some $\mathbf{k} = (k_i)_{i\notin\sigma} \in \mathbb{N}^{n-d}$. Since β is very generic then the negative support of $v^{\mathbf{k}}$ is the empty set and hence $\phi_{v^{\mathbf{k}}}$ is annihilated by $H_A(\beta)$. Moreover, the summation index set $N_{v^{\mathbf{k}}}$

in the series $\phi_{v^{\mathbf{k}}}$ is given by the integer vectors in an affine translate of the positive span of the columns of B_σ. The sum of the coordinates of the i-th column of B_σ is $1-|A_\sigma^{-1}a_{d+i}|$ where $|\ |$ means the sum of the corresponding coordinates. We have the following:

Theorem 2 [7, Theorem 3.11] *Under the above conditions the series $\phi_{v^{\mathbf{k}}}$ is a Gevrey solution of $M_A(\beta)$ along $Z=\{x_j=0:\ |A_\sigma^{-1}a_j|>1\}$ with index $s=\max\{|A_\sigma^{-1}a_j|:j=1,\dots,n\}$ at points in certain relatively open subset of Z. In particular, if $|A_\sigma^{-1}a_j|\leq 1$ for all $1\leq j\leq n$ then $\phi_{v^{\mathbf{k}}}$ is convergent.*

On the other hand, by Corollary 1, a slope $s>1$ of $M_A(\beta)$ along $Y=\{x_n=0\}$ corresponds to a number $s>1$ such that $\frac{1}{s}a_n$ belongs to the hyperplane H_τ supported on a facet τ of the convex hull of $\{0,a_1,\dots,a_{n-1}\}$ such that $0\notin\tau$. In particular, for any d-simplex $\sigma\subseteq\tau$ it is easy to check that $s=|A_\sigma^{-1}a_n|>1$. We say in this case that σ is a d-simplex corresponding to the slope $s>1$ of $M_A(\beta)$ along Y.

The following Theorem is a summary of some of the results in [7]. Its last statement uses results in [12] and in [17]. For the definition of a regular triangulation see, e.g., [19, Ch. 8].

Theorem 3 *Assume that $\beta\in\mathbb{C}^d$ is very generic and that $s>1$ is a slope of $M_A(\beta)$ along $Y=\{x_n=0\}$. For any d-simplex σ corresponding to s one can construct* $\mathrm{vol}(\sigma)=|\det(A_\sigma)|$ *many linearly independent Gevrey solutions $\phi_{v^{\mathbf{k}}}$ of $M_A(\beta)$ along Y with index s by varying $\mathbf{k}\in\mathbb{N}^{n-d}$ in a set Λ so that $\{A_{\bar\sigma}\mathbf{k}:\ \mathbf{k}\in\Lambda\}$ is a set of representatives of the group $\mathbb{Z}^d/\mathbb{Z}A_\sigma$. Moreover, if we repeat this construction for all the d-simplices σ corresponding to s which belong to a suitable regular triangulation for A and take the classes modulo $\mathcal{O}_{\widehat{X|Y}}(<s)$ we obtain a basis for the space of solutions of $M_A(\beta)$ in the space $(\mathcal{O}_{\widehat{X|Y}}(s)/\mathcal{O}_{\widehat{X|Y}}(<s))_p$ for points $p\in Y$ in a relatively open set of Y.*

We have exhibited the construction of the Gevrey solutions of $M_A(\beta)$ along $Y=\{x_n=0\}$ corresponding to each slope $s>1$ of $M_A(\beta)$ along Y for β very generic.

Let us construct Gevrey solutions of $M_A(\beta)$ at infinity. We will use the following notation: X' will be $\mathbb{C}^n$ with coordinates $(x_1,\dots,x_{n-1},z)$ and $z=1/x_n$ so that $X\cap X'=\mathbb{C}^{n-1}\times\mathbb{C}^*$. Denote $Y'=\{x_n=\infty\}=\{z=0\}\subseteq X'$.

Take $L_{-r}=F-rV$ where V is the V-filtration along Y. Notice that $\Phi_A^{L_{-r}}$ is not locally constant at $r=s-1>0$ if and only if $\frac{1}{(2-s)}a_n$ belongs to the hyperplane H_τ supported on a facet τ of the convex hull of $\{0,a_1,\dots,a_{n-1}\}$ such that $0\notin\tau$.

Theorem 4 *Assume that $\beta\in\mathbb{C}^d$ is very generic and that there exists $s>1$ such that $\frac{1}{(2-s)}a_n$ belongs to a hyperplane H_τ as above. For any d-simplex $\sigma\subseteq\tau$ one can construct* $\mathrm{vol}(\sigma)=|\det(A_\sigma)|$ *many linearly independent Gevrey series $\phi_{v^{\mathbf{k}}}$ along Y' with index s by varying $\mathbf{k}\in\mathbb{N}^{n-d-1}\times\mathbb{Z}_{<0}$ in a set Λ so that $\{A_{\bar\sigma}\mathbf{k}:\ \mathbf{k}\in\Lambda\}$ is a set of representatives of the group $\mathbb{Z}^d/\mathbb{Z}A_\sigma$. The classes of these series modulo convergent series $\mathcal{O}_{X'|Y'}$ are solutions of $M_A(\beta)$ in $\mathcal{O}_{\widehat{X'|Y'}}(s)/\mathcal{O}_{X'|Y'}$.*

Moreover, if we repeat this construction for all the d-simplices σ as above which belong to a suitable regular triangulation for A and take the classes modulo $\mathcal{O}_{\widehat{X'|Y'}}(<s)$ then we obtain a basis for the space of solutions of $M_A(\beta)$ in the space $(\mathcal{O}_{\widehat{X'|Y'}}(s)/\mathcal{O}_{\widehat{X'|Y'}}(<s))_p$ for points $p\in Y'$ in a relatively open set of Y'.

Proof.-

As before, by reordering the variables we can assume $\sigma = \{1, \ldots, d\}$ without loss of generality. Another basis of $\ker(A) = \{u \in \mathbb{Q}^n : \ Au = 0\}$ is given by the columns of the matrix:

$$B'_\sigma = \begin{pmatrix} -A_\sigma^{-1}a_{d+1} & -A_\sigma^{-1}a_{d+2} & \cdots & -A_\sigma^{-1}a_{n-1} & A_\sigma^{-1}a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Take a vector $v \in \mathbb{C}^n$ such that $Av = \beta$, $v_i \in \mathbb{N}$ for all $i \notin \sigma \cup \{n\}$ and $v_n \in \mathbb{Z}_{<0}$. It is clear that such a vector can be taken again as $v = v^{\mathbf{k}}$ but with $k_n \in \mathbb{Z}_{<0}$ and $k_i \in \mathbb{N}$ for $d+1 \leq i \leq n-1$. Moreover, the summation index set $N_{v^{\mathbf{k}}}$ in the series $\phi_{v^{\mathbf{k}}}$ is given by the integer vectors in an affine translate of the positive span of the columns of B'_σ. Notice that the sum of the coordinates of the i-th column of B'_σ is $1-|A_\sigma^{-1}a_{d+i}| \geq 0$ for $i = 1, \ldots, n-d-1$ while the sum of the coordinates of the last column is $|A_\sigma^{-1}a_n|-1 = 2-s-1 = 1-s < 0$. This implies that after the change $x_n^{-1} = z$ in the series $\phi_{v^{\mathbf{k}}}$ we have that $\phi'_{v^{\mathbf{k}}}(x_1, \ldots, x_{n-1}, z) = \phi_{v^{\mathbf{k}}}(x_1, \ldots, x_{n-1}, 1/x_n)$ is a Gevrey series along Y' with index $s = r+1 > 1$. In order to see that s is the Gevrey index of $\phi'_{v^{\mathbf{k}}}$ one can use a slightly modified version of Lemma 3.8 in [7].

On the other hand, since β is very generic then the negative support of $v^{\mathbf{k}}$ is $\text{nsupp}\,(v^{\mathbf{k}}) = \{n\}$ which is not minimal and hence $\phi_{v^{\mathbf{k}}}$ is not annihilated by $H_A(\beta)$. However, it can be checked that for any differential operator $P \in H_A(\beta)$ the series $P(\phi_{v^{\mathbf{k}}})$ is either zero or a polynomial in $z = x_n^{-1}$ with coefficients that are convergent power series in the variables $x_1, \ldots, x_{n-1}$. Thus, the proof of the Theorem finishes by using similar methods as in [7] and results from [18, Sec. 5] but with the differences that we have mentioned. Q.E.D.

Remark 2 *A slightly weaker version of the first paragraph of Theorem 4 can also be proven without the very genericity assumption in $\beta \in \mathbb{C}^d$. More precisely, for all $\beta \in \mathbb{C}^d$ if Φ_A^{L-r} is not locally constant at $r = s-1 > 0$ we can construct a Gevrey series along Y' of index $s = r+1$ which is a solution of $M_A(\beta)$ modulo the space $\mathcal{O}_{\widehat{X'|Y'}}(< s)$ (of Gevrey series along Y' with order less than s) by methods similar to the ones in [7, Section 4].*

We have the following corollary, which is a particular case of [18, Conjecture 5.18].

Corollary 2 *The real number $s > 1$ is a slope of $M_A(\beta)$ along $Y' = \{x_n = \infty\}$ if and only if Φ_A^{L-r} is not locally constant at $r = s-1$.*

Proof.- For the only if direction of the proof we refer to [18, Section 5]. Let us prove the if direction. By Theorem 4 and Remark 2 one can construct a Gevrey series of index $s = r+1$ that is a solution of $M_A(\beta)$ along Y' (modulo $\mathcal{O}_{\widehat{X'|Y'}}(< s)$). So, the result follows from the comparison theorem for the slopes [12]. Q.E.D.

Example 1 *Set*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

and $\beta \in \mathbb{C}^2$. *We have that the kernel of* A *is generated by* $u = (1, 1, -3)$ *and thus the hypergeometric ideal* $H_A(\beta)$ *is generated by* $\Box_u = \partial_1\partial_2 - \partial_3^3$, $E_1 - \beta_1 = 2x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - \beta_1$ *and* $E_2 - \beta_2 = x_1\partial_1 + 2x_2\partial_2 + x_3\partial_3 - \beta_2$. *Take* $\sigma = \{1, 2\}$ *and notice that for* $s = 4/3$ *we have that* $a_3/(2 - s)$ *belongs to the line* H_σ *determined by* a_1 *and* a_2. *By Corollary 2 we have that* $s = 4/3$ *is a slope of* $M_A(\beta)$ *along* $Y' = \{x_3 = \infty\}$. *Indeed, if we consider* $v^k = ((2\beta_1 - \beta_2 - k)/3, (2\beta_2 - \beta_1 - k)/3, k)$ *for* $k \in \mathbb{Z}_{<0}$ *we have that* ϕ_{v^k} *is a Gevrey series along* Y' *of index* $s = 4/3$ *if* $v_1^k, v_2^k \notin \mathbb{Z}_{<0}$. *Moreover, it is easy to check that* $(E_i - \beta_i)(\phi_{v^k}) = 0$ *for* $i = 1, 2$ *and for the three highest* $k \in \mathbb{Z}_{<0}$ *verifying that* $v_1^k, v_2^k \notin \mathbb{Z}_{<0}$ *we have that* $\Box_u(\phi_{v^k}) = v_1^k v_2^k x_1^{v_1^k - 1} x_2^{v_2^k - 1} x_3^k$, *which is convergent at any* $p \in Y' \cap \{x_1x_2 \neq 0\}$. *If fact the classes modulo* $\mathcal{O}_{X'|Y'}$ *of these three series form a basis for the space of solutions of* $M_A(\beta)$ *in* $(\mathcal{O}_{\widehat{X'|Y'}}(s)/\mathcal{O}_{X'|Y'})_p$, $p \in Y' \cap \{x_1x_2 \neq 0\}$. *Notice that if* β *is very generic then* $k = -1, -2, -3$ *above.*

4 Irregularity of modified A-hypergeometric systems

4.1 Fourier transform and initial ideals

Let D be the Weyl algebra $\mathbb{C}\langle x_1, \ldots, x_n, t, \partial_1, \ldots, \partial_n, \partial_t\rangle$. The variable t is also denoted by x_{n+1} and ∂_t by ∂_{n+1}.

Let $L : \mathbb{R}^{2n+2} \to \mathbb{R}$ be a linear form $L(\alpha, \beta) = \sum_i u_i\alpha_i + v_i\beta_i$ such that $u_i + v_i \geq 0$ for $i = 1, \ldots, n+1$ inducing the so-called L–filtration on the ring D. If $u_i + v_i > 0$ for all i, the associated graded ring $\mathrm{gr}^L(D)$ is isomorphic to a polynomial ring in $2n + 2$ variables $(x, \xi) = (x_1, \ldots, x_{n+1}, \xi_1, \ldots, \xi_{n+1})$ with complex coefficients. This polynomial ring is L-graded, the L-degree of a monomial $x^\alpha\xi^\beta$ being $L(\alpha, \beta)$. If we need to emphasize the coefficients of the linear form we will simply write $L = L_{(u,v)}$ for $(u, v) \in \mathbb{R}^{2n+2}$ with $u_i + v_i \geq 0$ for all i. If $u = \mathbf{0} \in \mathbb{N}^{n+1}$ and $v = \mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{N}^{n+1}$ then the corresponding $L_{(u,v)}$ filtration is nothing but the usual order filtration on D (which is also called the F-filtration). If $u = (\mathbf{0}, -1) \in \mathbb{N}^{n+1}$ and $v = -u \in \mathbb{N}^{n+1}$ then the corresponding $L_{(u,v)}$ filtration is nothing but the Malgrange-Kashiwara filtration on D (we also say the V-filtration) with respect to $t = 0$. In the remaining part of this Section we will assume $u_i + v_i > 0$ for all i; we will say then that (u, v) is a weight vector for the Weyl algebra.

We define the ring isomorphism $\mathcal{F}$ of D by $t \mapsto -\partial_t$, $\partial_t \mapsto t$. The isomorphism $\mathcal{F}$ is called the *Fourier transform* on D with respect to the variable t. The inverse transform $\mathcal{F}^{-1}$ is given by $t \mapsto \partial_t$, $\partial_t \mapsto -t$. Let (u, v) be a weight vector for the Weyl algebra. The Fourier transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ induce isomorphisms in $\mathrm{gr}^L(D)$ and we denote them also by $\mathcal{F}$ and $\mathcal{F}^{-1}$ respectively. Analogously, if $C \subset \mathbb{C}^{2n+2}$ is the affine algebraic set defined by an ideal $J \subseteq \mathrm{gr}^L(D)$ we write $\mathcal{F}C$ and $\mathcal{F}^{-1}C$ for the algebraic set defined by $\mathcal{F}J \subseteq \mathrm{gr}^L(D)$ and $\mathcal{F}^{-1}J \subseteq \mathrm{gr}^L(D)$ respectively.

We define the Fourier transform of the weight vector (u, v) by the formula $\mathcal{F}(u, v) := (u_1, \ldots, u_n, v_{n+1}, v_1, \ldots, v_n, u_{n+1})$. We notice that $\mathcal{F}\mathcal{F}(u, v) = (u, v)$. We will also write $\mathcal{F}L = \mathcal{F}(u, v)$ if $L = L_{(u,v)}$.

Proposition 1 *For any operator $\ell \in D$, we have*

$$\mathrm{in}_{(u,v)}(\ell) = \mathcal{F}^{-1}\mathrm{in}_{\mathcal{F}(u,v)}(\mathcal{F}\ell)$$

Proof. We prove it in the case $n = 0$. Other cases can be reduced to this case. We put $\xi_{n+1} = \mathrm{in}_{(u,v)}(\partial_t)$. We assume that $u + v > 0$ and $\ell = t^a \partial_t^b$. Then, we have $\mathrm{in}_{(u,v)}(\ell) = t^a \xi_{n+1}^b$. Since $\mathcal{F}\ell = (-\partial_t)^a t^b = (-1)^a(t^b \partial_t^a + abt^{b-1}\partial_t^{a-1} + \cdots)$ and $u + v > 0$, we have $\mathrm{in}_{(v,u)}(\mathcal{F}\ell) = (-1)^a t^b \xi_{n+1}^a$. Applying the inverse Fourier transform, we obtain $\mathcal{F}^{-1}\mathrm{in}_{(v,u)}(\mathcal{F}\ell) = (-1)^a \xi_{n+1}^b (-t)^a = \mathrm{in}_{(u,v)}(\ell)$.

Suppose that $t^a \partial_t^b >_{(u,v)} t^{a'} \partial_t^{b'}$. Then, we have $(-\partial_t)^a t^b >_{(v,u)} (-\partial_t)^{a'} t^{b'}$. Thus, we obtain the conclusion. Q.E.D.

Proposition 1 yields the following simple, but important claim for the Gröbner deformation method.

Corollary 3 *For any left ideal I in D, we have*

$$\mathrm{in}_{(u,v)}(I) = \mathcal{F}^{-1}\mathrm{in}_{\mathcal{F}(u,v)}(\mathcal{F}I)$$

Proof. Since $\mathrm{in}_{(u,v)}(I)$ is spanned by $\mathrm{in}_{(u,v)}(\ell)$, $\ell \in I$ as a $\mathbb{C}$-vector space, the conclusion follows from the previous proposition. Q.E.D.

4.2 Slopes of modified A-hypergeometric systems

We retain the notations of [20], which are explained in the introduction. We are interested in the slopes of the modified system $M_{A,w,\alpha}(\beta)$ at any point along $T = \{t = 0\}$. We notice that $H_{A,w,\alpha}(\beta)$ is the Fourier transform of $H_{\widetilde{A}(w)}(\tilde{\beta})$ where $\tilde{\beta} = (\beta, \alpha - 1)$, i.e.

$$M_{A,w,\alpha}(\beta) = \mathcal{F}M_{\widetilde{A}(w)}(\widetilde{\beta}). \tag{7}$$

Recall that $F = (\mathbf{0}, \mathbf{1})$, $V = (0, \ldots, 0, -1; 0, \ldots, 0, 1)$ and $L_r = F + rV = (0, \ldots, 0, -r, 1, \ldots, 1, 1+r)$. Thus $\mathcal{F}L_r = (0, \ldots, 0, 1 + r, 1, \ldots, 1, -r)$.

In order to obtain the slopes of the modified system $M_{A,w,\alpha}(\beta)$, we need to study the L_r–initial ideal of the modified hypergeometric ideal $H_{A,w,\alpha}(\beta)$, for $r \in \mathbb{R}_{>0}$. Therefore, applying Corollary 3 and (7), we have

$$\mathrm{in}_{L_r}(H_{A,w,\alpha}(\beta)) = \mathcal{F}^{-1}\mathrm{in}_{\mathcal{F}L_r}(H_{\widetilde{A}(w)}(\tilde{\beta})). \tag{8}$$

Using (8), we have

$$Ch^{L_r}(M_{A,w,\alpha}(\beta)) = \mathcal{F}^{-1}(Ch^{\mathcal{F}L_r}(M_{\widetilde{A}(w)}(\widetilde{\beta}))) \tag{9}$$

On the other hand, since $M_{\widetilde{A}(w)}(\widetilde{\beta})$ is a hypergeometric system and the matrix $\widetilde{A}(w)$ is pointed, Theorem 1 gives a description of the irreducible components of $Ch^{\mathcal{F}L_r}(M_{\widetilde{A}(w)}(\widetilde{\beta}))$ in terms of the $(\widetilde{A}(w), \mathcal{F}L_r)$–umbrella. We notice here that the last component of $\mathcal{F}L_r$ equals $-r < 0$. We will recall the definition [17, Def. 2.7] of the umbrella in this case. First of all let us recall *loc. cit.*, that if a, b are two points and H is a hyperplane in $\mathbb{P}^{d+1}(\mathbb{R})$ containing neither a nor b, then the convex hull $\mathrm{conv}_H(a,b)$ of a and b relative to H, is the unique line segment joining a and b and not meeting H.

Let us denote the $i-th$ column of $\widetilde{A}(w)$ by $\widetilde{a}_i$, say $\widetilde{a}_i = \binom{a_i}{w_i}$ for $i = 1,\dots,n$ and $\widetilde{a}_{n+1} = (0,\dots,0,1)^t$. For simplicity let us write $(\widetilde{A},\widetilde{L})$ instead of $(\widetilde{A}(w),\mathcal{F}L_r)$. We view $\widetilde{a}_1,\dots,\widetilde{a}_{n+1}$ as points in $\mathbb{R}^{d+1} \subset \mathbb{P}^{d+1}(\mathbb{R})$. As $\widetilde{A}$ is pointed, there exists a linear form h on $\mathbb{R}^{d+1}$ such that $h(\widetilde{a}_i) > 0$ for all i. Let $\epsilon \in \mathbb{R}$ be such that $0 < \epsilon < h(\widetilde{a}_i)$ for $i = 1,\dots,n$ and $0 < \epsilon < \frac{h(\widetilde{a}_{n+1})}{r}$.

Definition 4 [17, Def. 2.7] *The $(\widetilde{A},\widetilde{L})$–polyhedron $\Delta_{\widetilde{A}}^{\widetilde{L}}$ is the convex hull*

$$\Delta_{\widetilde{A}}^{\widetilde{L}} = \mathrm{conv}_{\mathrm{H}_\epsilon}(\{\mathbf{0},\widetilde{a}_1,\dots,\widetilde{a}_n,\frac{\widetilde{a}_{n+1}}{-r}\}) \subset \mathbb{P}^{d+1}(\mathbb{R})$$

where H_ϵ is the projective closure of the affine hyperplane $h^{-1}(-\epsilon)$. The $(\widetilde{A},\widetilde{L})$-umbrella $\Phi_{\widetilde{A}}^{\widetilde{L}}$ is the set of faces of $\Delta_{\widetilde{A}}^{\widetilde{L}}$ which do not contain the origin. In particular, $\Phi_{\widetilde{A}}^{\widetilde{L}}$ contains the empty face.

The following figures show two $(\widetilde{A}(w),\mathcal{F}L_r)$–umbrellas for $A = (1,4)$ and $w = (-1,1)$. In each case the shaded region is the polyhedron and the fat boundary is the umbrella.

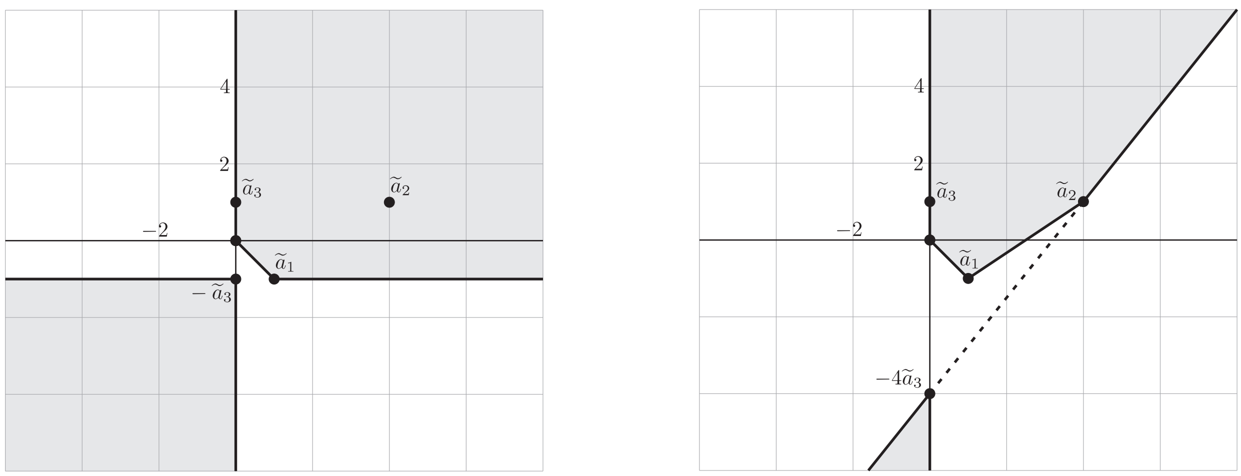

Figure 1: The first umbrella is for $r = 1$ and the second one is for $r = 1/4$.

From Figure 1 we can also see that for $r = 3/5$ the point $\frac{\widetilde{a}_3}{-r}$ belongs to the line passing through $\widetilde{a}_1$ and $\widetilde{a}_2$ which means that $s = r + 1 = 8/5$ is a slope of the system along $x_3 = 0$.

The following Corollary follows from Corollary 1 by using a slightly modified version of Remark 1.

Corollary 4 *Assume $s = r + 1 > 1$. Then s is a slope of $M_{A,w,\alpha}(\beta)$ along T at $p \in T$ if and only if $\Phi_{\widetilde{A}(w)}^{\mathcal{F}L_{r'}}$ is not locally constant at $r' = r$.*

Let us denote by A_w the matrix with columns $\widetilde{a}_i$, $1 \leq i \leq n$, and let Δ_{A_w} be the convex hull of $\{\widetilde{a}_i :\ 1 \leq i \leq n\}$ and the origin. With this notation, Corollary 4 can be rephrased as follows.

Corollary 5 *Assume $s = r+1 > 1$. Then s is a slope of $M_{A,w,\alpha}(\beta)$ along T at $p \in T$ if and only if there exists a facet τ of Δ_{A_w} such that $0 \notin \tau$ and $-\frac{1}{r}\widetilde{a}_{n+1} \in H_\tau$, where H_τ is the hyperplane that contains τ.*

Remark 3 *We notice that, by the definition of the modified system $M_{A,w,\alpha}(\beta)$, if we consider the change of variable $z = 1/t$ in $t \neq 0$ and then extend the D-module for $z \in \mathbb{C}$ in a natural way, we get the modified system $M_{A,-w,-\alpha}(\beta)$. Hence, the study of the irregularity of $M_{A,w,\alpha}(\beta)$ along $T = \{t = 0\}$ is equivalent to the study of the irregularity of $M_{A,-w,-\alpha}(\beta)$ along $T' = \{t = \infty\}$. As a consequence, Corollary 5 also provides a description of the slopes of the modified system $M_{A,w,\alpha}(\beta)$ along T' by using $\Delta_{A_{-w}}$ instead of Δ_{A_w}.*

Until the end of this section we will denote either $\Phi^v_{\widetilde{A}}$ or $\Phi^L_{\widetilde{A}}$ for the $(\widetilde{A}, L)$–umbrella with $L = L_{(u,v)}$ since Definition 4 does not depend on L but only on $v \in \mathbb{R}^{n+1}$. Moreover, for any subset $\eta \subseteq \{1,\ldots,n\}$ we denote by w_η the vector with coordinates equal to the ones of w indexed by η, i.e. $w_\eta = (w_i)_{i\in\eta}$.

Lemma 1 *For all real number $r > 0$ small enough we have that*

$$\{\eta' \in \Phi^{\mathcal{F}L_r,d}_{\widetilde{A}(w)} :\ n+1 \in \eta'\} = \{\sigma \cup \{n+1\} :\ \sigma \in \Phi^{w_\eta,d-1}_{A_\eta}, \eta \in \Phi^{F,d-1}_{A}\}$$

In particular, if all the facets in Φ^F_A have exactly d columns of A then the set of facets of $\Phi^{\mathcal{F}L_r,d}_{\widetilde{A}(w)}$ which contain $n+1$ is $\{\sigma \cup \{n+1\} : \sigma \in \Phi^{F,d-1}_A\}$.

Lemma 2 *For all $r > 0$ big enough we have that*

$$\Phi^{\mathcal{F}L_r,d}_{\widetilde{A}(w)} = \{\sigma \cup \{n+1\} : \sigma \in \Phi^{w,d-1}_A\}$$

Proposition 2 *The following conditions are equivalent:*

(a) $\Phi^w_A = \{\sigma \in \Phi^{w_\eta}_{A_\eta} :\ \eta \in \Phi^F_A\}$.

(b) $\Phi^{\mathcal{F}L_r}_{\widetilde{A}(w)}$ *is constant for all $r > 0$.*

(c) $M_{A,w,\alpha}(\beta)$ *does not have slopes along T.*

Proof. The equivalence between (b) and (c) is nothing but a direct consequence of Corollary 5. The equivalence between (a) and (b) follows from Lemma 1, Lemma 2 and the fact that any umbrella is determined by its facets. Q.E.D.

Remark 4 *Notice that condition (a) in Proposition 2 means that w induces a regular triangulation of A that refines the polyhedral complex subdivision induced by the weight vector $(1,\ldots,1)$. In other words, w is a perturbation of $(1,\ldots,1)$ for the matrix A. For example, this condition is satisfied if either the row span of the matrix A contains the vector $(1,\ldots,1)$ or w is the sum of a vector in the row span of the matrix A and a vector $(k,\ldots,k)$ with $k\geq 0$.*

Remark 5 *We would like to point out that even in the case when $M_{\widetilde{A}(w)}(\widetilde{\beta})$ is a regular holonomic hypergeometric system and the modified system $M_{A,w,\alpha}(\beta)$ has no slopes along T, the latter system is irregular if the original hypergeometric system $M_A(\beta)$ is. We illustrate this remark with Example 2.*

Example 2 *Take $A=(1\ 2)$ and $\beta\in\mathbb{C}$. The hypergeometric system $M_A(\beta)$ is irregular along $Y=\{x_2=0\}$ with unique slope $s=r+1=2$.*

If we choose $w=(1,1)$ and consider the matrix

$$\widetilde{A}(w)=\begin{pmatrix} 1 & 2 & 0\\ 1 & 1 & 1\end{pmatrix}$$

we have that the hypergeometric system $M_{\widetilde{A}(w)}(\beta,\alpha-1)$ is regular holonomic for all $\beta\in\mathbb{C}^d$ and $\alpha\in\mathbb{C}$ by a well known result of Hotta. However, the modified system $M_{A,w,\alpha}(\beta)$ has a slope $s=r+1=2$ along T' because $M_{A,-w,-\alpha}(\beta)$ has the slope $s=2$ along T (see Corollary 5 and Remark 3).

Remark 6 *Let φ be the map (4) defined in the introduction. Since the $\mathcal{D}$-modules $\mathcal{M}_{\widetilde{A}(\mathbf{0})}(\beta,-\alpha)$ and $\mathcal{M}_{A,w,\alpha}(\beta)$ are isomorphic when restricted to $\mathbb{C}^n\times\mathbb{C}^*$, the slopes of both modules along any coordinate subspace Y not contained in T coincide in $Y\setminus T$. Moreover, the map φ^* also induces an isomorphism for their spaces of Gevrey solutions along Y.*

4.3 Holomorphic solutions

We will study convergent and formal power series solutions of the modified A-hypergeometric module $\mathcal{M}_{A,w,\alpha}(\beta)$.

As said before, the map φ (4) induces an isomorphism between the $\mathcal{D}$-modules $\mathcal{M}_{\widetilde{A}(\mathbf{0})}(\beta,-\alpha)$ and $\mathcal{M}_{A,w,\alpha}(\beta)$ when restricted to $\mathbb{C}^n\times\mathbb{C}^*$, and also an isomorphism between the corresponding spaces of holomorphic solutions. More precisely, for any germ of holomorphic function $f(x,t)$ at a point (x_0,t_0) in X^*, the function $f(x,t)$ is a solution of $\mathcal{M}_{A,w,\alpha}(\beta)$ if and only if $\varphi^*(f)(y,s)=f(s^{-w_1}y_1,\ldots,s^{-w_n}y_n)$ is a germ of solution of $\mathcal{M}_{\widetilde{A}(\mathbf{0})}(\beta,-\alpha)$ at the point $(y_0,s_0)\in Y^*$ such that $\varphi(y_0,s_0)=(x_0,t_0)$. We can rewrite this as follows: the morphism

$$\varphi^*:\mathcal{H}om_{\mathcal{D}_{X^*}}(\mathcal{M}_{A,w,\alpha}(\beta)_{|X^*},\mathcal{O}_{X^*})\xrightarrow{\sim}\mathcal{H}om_{\mathcal{D}_{Y^*}}(\mathcal{M}_{\widetilde{A}(\mathbf{0})}(\beta,-\alpha)_{|Y^*},\mathcal{O}_{Y^*})$$

is an isomorphism of sheaves of vector spaces.

As a consequence the holonomic rank of both modules coincide

$$\operatorname{rank}(H_{A,w,\alpha}(\beta))=\operatorname{rank}(H_{\widetilde{A}(\mathbf{0})}(\beta,-\alpha))$$

and this last rank equals the one of $H_A(\beta)$ for any w, see [20, Theorem 1].

Recall that if β is generic $\operatorname{rank}(H_A(\beta))=\operatorname{vol}(A)$, where $\operatorname{vol}(A)$ is the normalized volume of the matrix A [8, 1, 16, 13], while in general $\operatorname{rank}(H_A(\beta))\geq\operatorname{vol}(A)$ [16, 13].

4.4 Gevrey solutions

We will describe the solutions of $\mathcal{M}_{A,w,\alpha}(\beta)$ in the space $\mathcal{O}_{\widehat{X|T}}$ of formal power series with respect to $T = \{t = 0\} \subset X = \mathbb{C}^{n+1}$. More generally, we also describe the solutions of $\mathcal{M}_{A,w,\alpha}(\beta)$ in the space $\sum_{\gamma\in\Lambda} t^{\gamma}\mathcal{O}_{\widehat{X|T}}$ for any finite set $\Lambda \subseteq \mathbb{C}$ (see Theorem 5).

We will assume in this Section that $w \in \mathbb{Z}^n$ and will freely use notations in [20]. Let τ be the weight vector $(\mathbf{0}, -1, \mathbf{0}, 1) \in \mathbb{Z}^{2n+2}$; we will take initial forms and initial ideals with respect to this weight vector that corresponds to the Malgrange-Kashiwara V–filtration along T.

Let us denote by $\widetilde{I}_{\widetilde{A}(w)} \subseteq \mathbb{C}[\partial, t] := \mathbb{C}[\partial_1, \ldots, \partial_n, t]$ the toric ideal associated with $\widetilde{A}(w)$, i. e. the binomial ideal generated by the operators in (3).

Lemma 3 *For all $w \in \mathbb{Z}^n$ we have $\operatorname{in}_{(\mathbf{0},-1)}(\widetilde{I}_{\widetilde{A}(w)}) = \mathbb{C}[\partial, t]\operatorname{in}_w(I_A)$.*

Proof. Recall that

$$\widetilde{I}_{\widetilde{A}(w)} = \langle \partial^{u_+} - \partial^{u_-} \mid Au = 0, w\cdot u = 0\rangle + \langle \partial^{u_+} - t^{w\cdot u}\partial^{u_-} \mid Au = 0, w \cdot u > 0\rangle$$

and we can write

$$I_A = \langle \partial^{u_+} - \partial^{u_-} \mid Au = 0, w\cdot u = 0\rangle + \langle \partial^{u_+} - \partial^{u_-} \mid Au = 0, w \cdot u > 0\rangle.$$

Notice that $\operatorname{in}_{(\mathbf{0},-1)}(\partial^{u_+} - t^{w\cdot u}\partial^{u_-}) = \partial^{u_+} = \operatorname{in}_w(\partial^{u_+} - \partial^{u_-})$ if $Au = 0$ and $w\cdot u = w\cdot u_+ - w\cdot u_- > 0$ and that $\operatorname{in}_{(\mathbf{0},-1)}(\partial^{u_+} - \partial^{u_-}) = \partial^{u_+} - \partial^{u_-} = \operatorname{in}_w(\partial^{u_+} - \partial^{u_-})$ if $Au = 0$ and $w \cdot u = 0$. The conclusion follows by a straightforward Groebner basis argument because a Groebner basis of $\widetilde{I}_{\widetilde{A}(w)}$ with respect to $(\mathbf{0}, -1)$ (resp. of I_A with respect to w) is given by a set of binomials with the same form as the ones defining the ideal. Q.E.D.

Recall that the indicial polynomial (also called b-function) of $H_{A,w}(\beta)$ along T is a polynomial $b(s) \in \mathbb{C}[s]$ such that $b(\theta_t)$ is the monic generator of $\operatorname{in}_\tau(H_{A,w}(\beta)) \cap \mathbb{C}[\theta_t]$ where $\theta_t = t\partial_t$. Moreover, we have by [20, Th. 3] that for β and w generic, the indicial polynomial of $H_{A,w}(\beta)$ along T is

$$b(s) = \prod_{(\partial^{\mathbf{k}},\sigma)\in\mathcal{T}(M)} (s - w\beta^{(\partial^{\mathbf{k}},\sigma)}) \tag{10}$$

where $M = \operatorname{in}_w(I_A)$, $\mathcal{T}(M)$ is the set of top-dimensional standard pairs of M and $v = \beta^{(\partial^{\mathbf{k}},\sigma)}$ is the vector determined by the equations $v_i = k_i \in \mathbb{N}$ for $i \notin \sigma$ and $Av = \beta$, which is also an exponent of $H_A(\beta)$ with respect to w (see [16, Lemma 4.1.3]).

Definition 5 *We say that a (generic) vector $\widetilde{w} \in \mathbb{Q}^n$ is a (generic) perturbation of $w \in \mathbb{Z}^n$, with respect to A, if there exists $w' \in \mathbb{Q}^n$ such that $\operatorname{in}_{\widetilde{w}}(I_A) = \operatorname{in}_{w'}(\operatorname{in}_w(I_A))$.*

Remark 7 *If $\widetilde{w}$ is generic we have that $\operatorname{in}_{\widetilde{w}}(I_A)$ is a monomial ideal and it is well known that the degree of a monomial ideal equals the cardinality of its set of top-dimensional standard pairs $\mathcal{T}(\operatorname{in}_{\widetilde{w}}(I_A))$. Moreover, for very generic $\beta \in \mathbb{C}^d$ there are exactly $\deg(\operatorname{in}_{\widetilde{w}}(I_A))$ many exponents of $H_A(\beta)$ with respect to $\widetilde{w}$; see [16, Sec. 3.4].*

Lemma 4 *Let $\beta \in \mathbb{C}^d$ be very generic and $w \in \mathbb{Z}^n$. There is a generic perturbation $\widetilde{w} \in \mathbb{Q}^n$ of w such that for any exponent $v \in \mathbb{C}^n$ of $H_A(\beta)$ with respect to $\widetilde{w}$ the series $\psi_v(x,t) = t^{-\alpha}\phi_v(t^{w_1}x_1, \ldots, t^{w_n}x_n)$ is a solution of $\mathcal{M}_{A,w,\alpha}(\beta)$ of the form $\psi_v(x,t) = \sum_{m\geq 0} f_m(x)t^{\gamma+m} \in t^\gamma \mathcal{O}_{\widehat{X|T},(p,0)}$, with $\gamma = wv - \alpha$ and $f_0(x) \neq 0$ for some $p \in \mathbb{C}^n$.*

Proof. Since β is very generic, for any generic $\widetilde{w}$ an exponent v of $H_A(\beta)$ with respect to $\widetilde{w}$ can be written as $v = \beta^{(\partial^{\mathbf{k}},\sigma)}$ where $(\partial^{\mathbf{k}},\sigma)$ is a top-dimensional standard pair of $\mathrm{in}_{\widetilde{w}}(I_A)$, see [16, Sec. 3.4]. In particular, $\sigma \in \Phi_A^{\widetilde{w},d-1}$, $\mathbf{k} = (k_i)_{i\notin\sigma} \in \mathbb{N}^{n-d}$, $Av = \beta$ and $v_i = k_i \in \mathbb{N}$ for all $i \notin \sigma$.

The series $\phi_v(x)$ is a Gevrey solution of $M_A(\beta)$ along a coordinate subspace $Z \subseteq \mathbb{C}^n$ (see Theorem 2) but since v is here an exponent we have that $N_v = \mathbb{N}B_\sigma \cap \mathbb{Z}^n$. A first consequence is that $f(x,t) = t^{-wv}\phi_v(t^{w_1}x_1, \ldots, t^{w_n}x_n) = t^{\alpha - wv}\psi_v(x,t)$ is a Gevrey solution of $M_{A,w,wv}(\beta)$ along $Z \times \mathbb{C} \subseteq \mathbb{C}^{n+1}$. It is enough to see that f is also a formal series along T. In order to see that we can write $f(x,t) = \sum_{m\geq 0} f_m(x)t^m \in \mathcal{O}_{\widehat{X|T},p}$ and recalling the expression of ϕ_v (6) it is enough to prove that for all $u \in N_v \setminus \{0\}$ we have $wu \in \mathbb{N}$ and that the coefficient of t^m in f, i. e.

$$f_m(x) = \sum_{u \in N_v, wu = m} \frac{[v]_{u_-}}{[v+u]_{u_+}} x^{v+u},$$

is a convergent series at some $p \in \mathbb{C}^n$.

We can take a generic perturbation $\widetilde{w} \in \mathbb{Q}^n$ of w of the form $\widetilde{w} = w + \epsilon\widetilde{e}$ with $\widetilde{e} = (1, \ldots, 1) + \epsilon' w'$ for $\epsilon > 0$ and $\epsilon' > 0$ small enough and $w' \in \mathbb{Q}^n$ is generic.

Since v is an exponent of $H_A(\beta)$ with respect to $\widetilde{w}$ we have by [16, (3.30)] that $\widetilde{w}u > 0$ for all $u \in N_v \setminus \{0\}$. Equivalently, we have that $\widetilde{w}u > 0$ for any column u of B_σ. Hence, since last inequality holds for $\epsilon > 0$ and $\epsilon' > 0$ small enough we have that $wu \geq 0$ for all $u \in N_v = \mathbb{N}B_\sigma \cap \mathbb{Z}^n$. Notice that when $wu > 0$ for all $u \in N_v \setminus \{0\}$ then the set $\{u \in N_v, wu = m\}$ is finite and hence $f_m(x)$ is clearly convergent. In general, since $N_v = \mathbb{N}B_\sigma \cap \mathbb{Z}^n$ we have that f_m is convergent if and only for any column u of B_σ such that $wu = 0$ we have that $|u| = |u_+| - |u_-| \geq 0$. Notice that $wu = 0$ implies $0 < \widetilde{w}u = \epsilon(|u| + \epsilon' w' u)$ and so $|u| + \epsilon' w' u > 0$. Hence, since this holds for $\epsilon' > 0$ small enough, we have that $|u| \geq 0$. We have proved that f is a formal solution of $M_{A,w,wv}(\beta)$ along T and it is clear that $f_0(x) = x^v + \cdots \neq 0$. Q.E.D.

Let $\widetilde{w} \in \mathbb{Q}^n$ be a generic perturbation of $w \in \mathbb{Z}^n$ as in the proof of Lemma 4.

Lemma 5 *If $f(x,t) = \sum_{m\geq 0} f_m(x)t^{\gamma+m} \in t^\gamma \mathcal{O}_{\widehat{X|T},p}$ is a solution of $\mathcal{M}_{A,w,\alpha}(\beta)$ for some $\gamma \in \mathbb{C}$, $p \in T$, with $f_0(x) \neq 0$, then:*

(a) $t^\alpha f(x,t)$ is a solution of $\mathcal{M}_{A,w}(\beta)$.

(b) For all $m \geq 0$, $f_m(x)$ is a holomorphic solution of $\mathcal{M}_{A_w}(\beta, \alpha+\gamma+m)$, where this last module is the hypergeometric system associated with the matrix A_w and the parameter $(\beta, \alpha+\gamma+m)$.

(c) $b(\alpha+\gamma) = 0$, where $b(s)$ is the indicial polynomial of $H_{A,w}(\beta)$ along T.

(d) If β is very generic, $\alpha+\gamma = wv$ for some exponent v of $H_A(\beta)$ with respect to $\widetilde{w}$.

Proof. The proof of (a) and (b) are straightforward. Let us prove (c). By (a) and using [16, Theorem 2.5.5] we have that $\mathrm{in}_{(0,1)}(t^\alpha f(x,t)) = f_0(x)t^{\alpha+\gamma}$ is a solution of $\mathrm{in}_\tau(H_{A,w}(\beta))$.

Recall by definition of $b(s)$ that $\langle b(\theta_t)\rangle = \mathrm{in}_\tau(H_{A,w}(\beta)) \cap \mathbb{C}[\theta_t]$ where $\theta_t = t\partial_t$. Thus, the differential operator $b(\theta_t)$ annihilates $\mathrm{in}_{(0,1)}(t^\alpha f(x,t)) = f_0(x)t^{\alpha+\gamma}$. This implies, using for example [16, Lemma 1.3.2], that $0 = b(\theta_t)(f_0(x)t^{\alpha+\gamma}) = b(\alpha+\gamma)f_0(x)t^{\alpha+\gamma}$ and this implies that $b(\alpha+\gamma) = 0$.

Let us proof (d). By (b) we have that $f_0(x)$ is a holomorphic solution of $H_{A_w}(\beta, \alpha+\gamma)$ and thus it can be written as a Nilsson series at the origin with respect to a vector $\widetilde{e}$ that is a perturbation of $e = (1,\ldots,1)$ (see [16], [15], [5]) and, in particular, it makes sense to consider the initial form of $f_0(x)$ with respect to $\widetilde{e}$. On the other hand, using Lemma 3, we have that $\mathrm{in}_w(I_A) + \langle A\theta - \beta\rangle \subseteq \mathrm{in}_\tau(H_{A,w}(\beta))$ annihilates $f_0(x)$. Since β is very generic, $\mathrm{in}_w(I_A) + \langle A\theta - \beta\rangle = \mathrm{in}_{(-w,w)} H_A(\beta)$. This implies that $f_0(x)$ is a solution of $\mathrm{in}_{(-w,w)} H_A(\beta)$ and hence, $\mathrm{in}_{\widetilde{e}}(f_0(x))$ is a solution of $\mathrm{in}_{(-\widetilde{w},\widetilde{w})} H_A(\beta)$ for $\widetilde{w} = w + \epsilon\widetilde{e}$. Thus, since β is very generic $\mathrm{in}_{\widetilde{e}}(f_0(x)) = cx^v$ for $c \in \mathbb{C}$ and v an exponent of $H_A(\beta)$ with respect to $\widetilde{w}$. Hence, using (b), we also have that $wv = \alpha+\gamma$. Q.E.D.

Remark 8 *Although we assume in the paper that $\widetilde{A}(w)$ is pointed it turns out that in this Section this fact is only used in the proof of (d) in Lemma 5. However, let us notice that if $\widetilde{A}(w)$ is not pointed and w is generic then $\mathrm{in}_w(I_A) = \mathbb{C}[\partial]$, $\mathrm{in}_\tau(H_{A,w}(\beta)) = D$ and so $b(s) = 1$. Thus, by (c) in Lemma 5 the modified system $\mathcal{M}_{A,w,\alpha}(\beta)$ does not have any solution in $t^\gamma \mathcal{O}_{\widehat{X|T},p}$ for all $\gamma \in \mathbb{C}$ and $p \in T$.*

Remark 9 *Since $\widetilde{w} = w + \epsilon\widetilde{e}$ with $\widetilde{e} = (1,\ldots,1) + \epsilon' w'$ for $\epsilon > 0$ and $\epsilon' > 0$ small enough we have that $\mathrm{in}_{\widetilde{w}}(I_A) = \mathrm{in}_{w'}(\mathrm{in}_e(\mathrm{in}_w(I_A)))$ for $e = (1,\ldots,1)$. In particular, $\mathrm{in}_{\widetilde{w}}(I_A)$ and $\mathrm{in}_w(I_A)$ have the same degree.*

Let us denote by $\dim_{\mathbb{C}}(M,\mathcal{F})_p$ the dimension of the space of $\mathcal{F}$-solutions of a D-module M at a point p.

Theorem 5 *Assume $\beta \in \mathbb{C}^d$ is very generic, $w \in \mathbb{Z}^n$ and $\alpha \in \mathbb{C}$. Then we have that $\dim_{\mathbb{C}}(\mathcal{M}_{A,w,\alpha}(\beta), \mathcal{O}_{\widehat{X|T}})_p = 0$ if $wv - \alpha \notin \mathbb{N}$ for all the exponents v of $H_A(\beta)$ with respect to $\widetilde{w}$. We also have that*

$$\dim_{\mathbb{C}}(\mathcal{M}_{A,w,\alpha}(\beta), \sum_{b(\alpha+\gamma)=0} t^\gamma \mathcal{O}_{\widehat{X|T}})_p = \deg(\mathrm{in}_w(I_A))$$

In particular, if w is generic we also have

$$\dim_{\mathbb{C}}(\mathcal{M}_{A,w,wv-m}(\beta), \mathcal{O}_{\widehat{X|T}})_p = 1$$

for all generic $p \in T$, $m \in \mathbb{N}$ and any exponent v of $H_A(\beta)$ with respect to w.

Proof. The first statement follows from Lemma 5, (d).

The inequality $\dim_{\mathbb{C}}(\mathcal{M}_{A,w,\alpha}(\beta), \sum_{b(\alpha+\gamma)=0} t^{\gamma}\mathcal{O}_{\widehat{X|T}})_p \geq \deg(\mathrm{in}_w(I_A))$ follows from Lemma 4, Remark 7 and Remark 9. On the other hand, since the differential operators defining $\mathcal{M}_{A,w,\alpha}(\beta)$ belong to the Weyl Algebra we have that any solution $f \in \sum_{b(\alpha+\gamma)=0} t^{\gamma}\mathcal{O}_{\widehat{X|T}}$ of $\mathcal{M}_{A,w,\alpha}(\beta)$ decomposes as a finite sum of solutions, each of them in a space $t^{\gamma}\mathcal{O}_{\widehat{X|T},p}$. Recall by the proof of Lemma 5 that any solution $f \in t^{\gamma}\mathcal{O}_{\widehat{X|T},p}$ of $\mathcal{M}_{A,w,\alpha}(\beta)$ verifies that $\mathrm{in}_{(\widetilde{w},0)}(\mathrm{in}_{(\mathbf{0},1)}(f)) = cx^v t^{wv-\alpha}$ for an exponent of $H_A(\beta)$ with respect to $\widetilde{w}$. This last fact, Remark 7, Remark 9 and a slightly modified version of [16, Proposition 2.5.7] prove that $\dim_{\mathbb{C}}(\mathcal{M}_{A,w,\alpha}(\beta), \sum_{b(\alpha+\gamma)=0} t^{\gamma}\mathcal{O}_{\widehat{X|T}})_p \leq \deg(\mathrm{in}_w(I_A))$.

Finally, the last statement follows from the second statement and from the fact that if β is very generic, $w \in \mathbb{Z}^n$ is generic and v and v' are two different exponents of $H_A(\beta)$ with respect to w, then $w(v-v') \notin \mathbb{Z}$. Q.E.D.

Remark 10 *Let $\psi_v(x,t)$ be the series constructed in Lemma 4 and used in Theorem 5. If w is in the row span of A then $f(x,t) = t^{\alpha-wv}\psi_v$ does not depend on t and thus it is a convergent series. If w is not in the row span of A then $f(x,t)$ is Gevrey along T with index $s = r+1$ where*

$$r = \max\{-\frac{|u|}{wu} \, : \, u \in N_v, wu > 0\}$$

where $|u| = \sum_i u_i$. On the other hand, as mentioned in the proof of Lemma 4 since β is very generic and v is an exponent of $H_A(\beta)$ with respect to $\widetilde{w}$ then v is associated with a simplex $\sigma \in \Phi_A^{\widetilde{w},d-1}$ and there is a basis $\{b_i \, : \, i \notin \sigma\}$ of the kernel of A such that for all $i \notin \sigma$, $(b_i)_j = 0$ for all $j \notin \sigma \cup \{i\}$ and $(b_i)_i = 1$. The set $\{b_i \, : \, i \notin \sigma\}$ is the set of columns of of B_σ (if we reorder the variables so that $\sigma = \{1,\ldots,d\}$) and in this case we have that $N_v = \mathbb{N}B_\sigma \cap \mathbb{Z}^n$. Thus, more explicitly, $r = \max\{-|b_i|/(wb_i) \, : \, i \notin \sigma, wb_i > 0\}$ where $\{b_i \, : \, i \notin \sigma\}$ is the set of columns of B_σ, $|b_i| = 1 - |A_\sigma^{-1}a_i|$ and $wb_i = w_i - w_\sigma A_\sigma^{-1}a_i > 0$. The proof of this formula is technical and follows from standard stimates on Gamma functions similar to the ones used in [7] to compute the index of Gevrey solutions for hypergeometric systems. In particular, if w is a perturbation of $(1,\ldots,1)$ then r is close to -1 and if A is homogeneous then $r = 0$ because $|u| = 0$ for any $u \in N_v$ and hence in both cases the series is convergent.

4.5 Gevrey solutions modulo convergent series

As pointed out by Theorem 5, if both $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}^d$ are very generic then $\mathcal{M}_{A,w,\alpha}(\beta)$ does not have any nonzero solution in $\mathcal{O}_{\widehat{X|T},p}$ for all $p \in T$. This is in contrast with the case of the irregularity of hypergeometric systems along coordinate hyperplanes, where for any slope $s = r+1$ of $\mathcal{M}_A(\beta)$ along $Y = \{x_n = 0\}$ and for very generic $\beta \in \mathbb{C}^d$ one can construct a formal solution ϕ of $\mathcal{M}_A(\beta)$ along Y, i.e. $\phi \in \mathcal{O}_{\widehat{X|Y},p}$, such that ϕ has Gevrey index equal to the algebraic slope (see [7]).

However, by the comparison theorem for the slopes [12] and the perversity of the irregularity complex of a holonomic D-module along a smooth hypersurface [14], one knows that for each algebraic slope $s = r+1$ of $\mathcal{M}_{A,w,\alpha}(\beta)$ along T at a generic $p \in T$ there must exist a formal series $\phi \in \mathcal{O}_{\widehat{X|T},p}$ with Gevrey index $s = r+1$ such that $P(\phi)$ is convergent at p for all $P \in H_{A,w,\alpha}(\beta)$.

The purpose of this section is to describe Gevrey solutions modulo convergent series of the modified system $M_{A,w,\alpha}(\beta)$ along T when $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}^d$ are very generic. To this end, we will use the construction of the Gevrey solutions at infinity of the hypergeometric system $M_{\widetilde{A}(w)}(\beta, \alpha - 1)$ as performed in Theorem 4.

Take $X' = \mathbb{C}^{n+1}$ with coordinates $(x_1, \ldots, x_n, z)$ and $z = 1/t$ so that $X \cap X' = \mathbb{C}^n \times \mathbb{C}^*$. Denote $T' = \{t = \infty\} = \{z = 0\} \subseteq X'$. We can consider for any $\gamma \in \mathbb{C}$ the morphism of vector spaces

$$\begin{array}{rcl} \Upsilon_\gamma : t^\gamma \mathcal{O}_{\widehat{X|T},p} & \longrightarrow & t^{-1-\gamma}\mathcal{O}_{\widehat{X'|T'},p'} \\ f = \sum_{m\geq 0} f_m(x) t^{\gamma+m} & \longmapsto & \Upsilon_\gamma(f) = \sum_{m\geq 0} f_m(x)(-\gamma-1)_m t^{-1-\gamma-m} \end{array}$$

where $p = (p_1, \ldots, p_n, 0) \in T$ and $p' = (p_1, \ldots, p_n, \infty) \in T'$.

Remark 11 *Notice that Υ_γ is an isomorphism if and only if $\gamma \notin \mathbb{Z}_{<0}$. In such a case we also have that $\Upsilon_\gamma(t^\gamma \mathcal{O}_{\widehat{X|T}}(s-1)) = t^{-1-\gamma}\mathcal{O}_{\widehat{X'|T'}}(s)$ for all s. It is also clear that $\Upsilon_0(\sum_{m\geq 0} f_m(x) t^{k+m}) = (-1)_k \Upsilon_k(\sum_{m\geq 0} f_m(x) t^{k+m})$ for all $k \in \mathbb{N}$.*

Theorem 6 *Assume $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}^d$ to be very generic. If $s = r+1 > 1$ is a slope of $\mathcal{M}_{A,w,\alpha}(\beta)$ along T then we can construct $\sum_\tau \operatorname{vol}(\operatorname{conv}(0, \widetilde{a}_i : i \in \tau))$ Gevrey series that are linearly independent solutions of $\mathcal{M}_{A,w,\alpha}(\beta)$ modulo convergent series and whose Gevrey index is equal to $s = r+1$. Here τ runs between all the facets of Δ_{A_w} such that $-\frac{1}{r}\widetilde{a}_{n+1} \in H_\tau$ and $0 \notin \tau$. Moreover, the classes of these Gevrey series modulo $\mathcal{O}_{\widehat{X|T}}(<s)$ form a basis of the solution space of $\mathcal{M}_{A,w,\alpha}(\beta)$ in $(\mathcal{O}_{\widehat{X|T}}(s)/\mathcal{O}_{\widehat{X|T}}(<s))_p$ for points $p \in T$ in a relatively open set of T.*

Proof. The existence of such facets τ is given by Corollary 5.

Since $-\frac{1}{r}\widetilde{a}_{n+1} \in H_\tau$ and $-r = 2 - s'$ for $s' = s+1 > 2$ we have that $s' > 2$ is a slope of $\mathcal{M}_{\widetilde{A}(w)}(\beta, \alpha-1)$ along $T' = \{t = \infty\}$. Thus, by Theorem 4 we can construct $\sum_\tau \operatorname{vol}(\operatorname{conv}(0, \widetilde{a}_i : i \in \tau)$ Gevrey series along $T' = \{t = \infty\}$ with index s'. Moreover, the classes in $\mathcal{O}_{\widehat{X'|T'}}(s')/\mathcal{O}_{\widehat{X'|T'}}(<s')$ are linearly independent solutions of the $\widetilde{A}(w)$-hypergeometric system with parameter $(\beta, \alpha - 1)$.

More precisely, for any $(d+1)$-simplex $\sigma \subseteq \tau$ the series constructed are of the form $\phi_{\widetilde{v}}$ for $\widetilde{v} = (v, -1-k)$ with $Av = \beta$, $wv - 1 - k = \alpha - 1$ (i. e., $wv - \alpha = k \in \mathbb{N}$) and $v_i \in \mathbb{N}$ for all $i \in \{1, \ldots, n\} \setminus \sigma$.

Using Remark 11 we can take $\psi_v(x,t)$ as the unique Gevrey series along T with index $s = r+1$ verifying $\Upsilon_0(\psi_v(x,t)) = \phi_{\widetilde{v}}$ for $\widetilde{v} = (v, -1-k)$.

We conclude by Remark 11 that the $\sum_\tau \operatorname{vol}(\operatorname{conv}(0, \widetilde{a}_i : i \in \tau)$ series $\psi_v(x,t)$ constructed are Gevrey series with index $s = s' - 1 = r + 1$ whose classes modulo $\mathcal{O}_{\widehat{X|T}}(<s)$ are linearly independent. Moreover, it can be checked that they are solutions of the modified system modulo $\mathcal{O}_{X|T}$ by using the fact that their images by the morphism Υ_0 are solutions of $\mathcal{M}_{\widetilde{A}(w)}(\beta, \alpha - 1)$.

Last statement follows from (7), [12] and [17]. Q.E.D.

Example 3 *Take $A = (1\ 3\ 5)$, $w = (0,1,1)$ and $\beta, \alpha \in \mathbb{C}$. We have that $\widetilde{I}_{\widetilde{A}(w)} = \langle \partial_2 - t\partial_1^3, \partial_3 - t\partial_1^5 \rangle$ and $H_{A,w,\alpha}(\beta) = D\widetilde{I}_{\widetilde{A}(w)} + D\langle x_1\partial_1 + 3x_2\partial_2 + 5x_3\partial_3 - \beta, x_2\partial_2 + x_3\partial_3 - t\partial_t - \alpha\rangle$. The unique slope of $M_{A,w,\alpha}(\beta)$ along T is $s = r+1 = 5$ since $-\frac{1}{4}\widetilde{a}_4$ belongs to the line passing through $\widetilde{a}_1, \widetilde{a}_3$ and $\sigma = \{1,3\}$ is a facet of $\Phi^{\mathcal{F}L_r}_{\widetilde{A}(w)}$ if and only if $r \geq 4$.*

The volume of σ is one and following the proofs of Theorem 6 and Theorem 4 we can take $v = (\beta - 5\alpha, 0, \alpha)$ which satisfies the conditions $wv - \alpha = k = 0 \in \mathbb{N}$, $v_2 = 0 \in \mathbb{N}$ and $Av = \beta$. We get that the series

$$\psi_v(x,t) = \sum_{m_2, m_2+m_3 \geq 0} \frac{[\beta - 5\alpha]_{3m_2+5m_3}}{[\alpha + m_3]_{m_3} m_2!} x_1^{\beta-5\alpha-3m_2-5m_3} x_2^{m_2} x_3^{\alpha+m_3} t^{m_2+m_3}$$

is a Gevrey solution (modulo convergent series) of $M_{A,w,\alpha}(\beta)$ along T with index $s = r+1 = 5$.

5 Borel transformation and asymptotic expansion

We assume that the row span of the matrix A does not contain the vector $(1,1,\dots,1)$, but A_w contains $(1,1,\dots,1)$ where A_w is the matrix $\begin{pmatrix} a_1 & \cdots & a_n \\ w_1 & \cdots & w_n \end{pmatrix}$. This case holds if and only if the weight vector w is in the image of $\bar{A}^T$ where $\bar{A}$ is the matrix $\begin{pmatrix} a_1 & \cdots & a_n \\ 1 & \cdots & 1 \end{pmatrix}$ and is not in the row span of A. Solutions of this case can be analyzed by utilizing the Borel transformation and the Laplace transformation.

We review here some basics of the Borel summation method which we require in the following (see [3] for the details). Let

$$f(z) = \sum_{\ell=0}^{\infty} f_\ell t^{\ell+\gamma} \in t^\gamma \mathbb{C}[[t]] \tag{11}$$

be a formal power series, where γ is a complex number satisfying $\Re\gamma > 0$ (this condition will be relaxed later. See (18).). Solutions constructed in Theorem 5 are of this form. If its coefficients satisfy

$$\left| f_\ell \right| \leq CK^\ell \Gamma(1 + (\ell+\gamma)/\kappa) \qquad (\ell = 0,1,2,\cdots) \tag{12}$$

with some positive constants C, K, κ, then the formal Borel transform (with index κ) defined by

$$\hat{\mathcal{B}}_\kappa[f](\tau) := \sum_{\ell=0}^{\infty} \frac{f_\ell}{\Gamma(1+(\ell+\gamma)/\kappa)} \tau^{\ell+\gamma}$$

is a convergent power series near $\tau = 0$. In addition to (12), if

(i) the function $\hat{\mathcal{B}}_\kappa[f]$ can be analytically continued to a sector

$$S(\theta,\delta) := \{re^{i\theta'};\ \left|\theta' - \theta\right| < \delta/2,\ r > 0\}$$

of infinite radius in a direction $\theta \in \mathbb{R}$ with an opening angle $\delta > 0$,

(ii) the analytic continuation of $\hat{\mathcal{B}}_\kappa[f]$ satisfies the growth estimate

$$\left|\hat{\mathcal{B}}_\kappa[f](\tau)\right| \le c_1 \exp\left[c_2\left|\tau\right|^\kappa\right] \tag{13}$$

in $S(\theta,\delta)$ with some positive constants $c_1, c_2 > 0$,

then we say f is κ-summable in a direction θ, and define the κ-sum (or the Borel sum with index κ) of f by the Laplace transformation

$$\mathcal{S}[f](t) = \mathcal{L}^\theta_\kappa \hat{\mathcal{B}}_\kappa[f](t) := \int_0^{e^{i\theta}\cdot\infty} e^{-(\tau/t)^\kappa} \hat{\mathcal{B}}_\kappa[f](\tau) d(\tau/t)^\kappa, \tag{14}$$

where

$$d(\tau/t)^\kappa = \frac{\kappa\tau^{\kappa-1}}{t^\kappa} d\tau.$$

There are two remarks here.

(a) Because of the growth condition (ii) of $\hat{\mathcal{B}}_\kappa[f]$, the Laplace integral (14) converges if t satisfies

$$\Re\left[\left(\frac{\tau}{t}\right)^\kappa\right] - c_2|\tau|^\kappa > 0. \tag{15}$$

Since

$$\Re\left[\left(\frac{\tau}{t}\right)^\kappa\right] - c_2|\tau|^\kappa = \left|\frac{\tau}{t}\right|^\kappa \left\{\cos\kappa(\theta - \arg t) - c_2|t|^\kappa\right\}$$

(note that $\arg\tau = \theta$), the Laplace integral (14) converges in

$$\{t;\, \cos\left[\kappa(\arg t - \theta)\right] > c_2|t|^\kappa\}. \tag{16}$$

The region (16) has infinitely many connected components. Here and in what follows we specify one of them by imposing $|\arg t - \theta| < \pi/(2\kappa)$. Since we can vary $\arg\tau$ in (14) slightly, we conclude that the Borel sum defines a holomorphic function in

$$\bigcup_{|\theta'-\theta|<\delta/2} \{t;\, \cos\left[\kappa(\arg t - \theta')\right] > c_2|t|^\kappa,\ |\arg t - \theta'| < \pi/(2\kappa)\}.$$

Therefore we can find $\rho > 0$ and $\alpha > \pi/\kappa$ such that the Borel sum of f is holomorphic in $S(\theta,\alpha,\rho) := S(\theta,\alpha) \cap \{z;\, 0 < |z| < \rho\}$. (cf. [3, the first paragraph of §2.1])

(b) If f is κ-summable in a direction θ, then f is a Gevrey asymptotic expansion of its Borel sum: For any closed subsector $\overline{S}$ of $S(\theta,\alpha,\rho)$, there exists $C', K' > 0$ for which the inequality

$$\left|t^{-\gamma}\mathcal{S}[f](t) - \sum_{n=0}^{N-1} f_\ell\, t^\ell\right| \le C'(K')^N \left|t\right|^N \Gamma(1 + N/\kappa), \tag{17}$$

for $N = 1, 2, 3, \cdots$, holds in $\overline{S}$. This follows from the estimates (13) and the relation

$$t^{\ell+\gamma} = \int_0^{e^{i\theta}\cdot\infty} e^{-(\tau/t)^\kappa} \frac{\tau^{\ell+\gamma}}{\Gamma(1 + (\ell+\gamma)/\kappa)} d(\tau/t)^\kappa \quad (= \mathcal{L}^\theta_\kappa \hat{\mathcal{B}}_\kappa[t^{\ell+\gamma}]).$$

See Theorem 1 in p.14 of [3] and its proof for the details. This property is also known to be a necessary and sufficient condition of the conditions (i) and (ii) stated above ([3, Theorem 1 (p.23)]).

The Borel summation method can be applicable to the case when

$$\ell + \gamma \notin -\kappa\mathbb{N} \quad \text{for} \quad \ell = 0, 1, 2, \ldots. \tag{18}$$

We can define the Borel transform $\hat{\mathcal{B}}_\kappa[f]$ in exactly the same manner. In this case, however, the Laplace integral (14) may not converge at $\tau = 0$. Therefore we modify the definition of the Borel sum to

$$\mathcal{S}[f](t) = \mathcal{L}^\theta_\kappa \hat{\mathcal{B}}_\kappa[f](t) := \frac{1}{1 - e^{-2\pi i\gamma/\kappa}} \int_{\Gamma_{\kappa\theta}} e^{-\zeta/t^\kappa} \hat{\mathcal{B}}_\kappa[f](\zeta^{1/\kappa}) \frac{d\zeta}{t^\kappa} \tag{19}$$

with a path of integration $\Gamma_{\kappa\theta}$ which runs from ∞ along $\arg\zeta = \kappa\theta - 2\pi$ to some point near the origin, takes a 2π radian turn along a circle with the center at the origin, and goes back to infinity in the direction $\arg\zeta = \kappa\theta$. When γ/κ is not an integer and satisfies $\Re\gamma > 0$, (19) coincides with (14). The Borel sum (19) also satisfies the same properties with the previous one (14).

We apply the Borel summation method to the Gevrey solution constructed in Theorem 5. Our main claim of this section is

Theorem 7 *We assume that the row span of the matrix A does not contain the vector $(1, 1, \ldots, 1)$, but that of A_w contains $(1, 1, \ldots, 1)$. Let*

$$\psi(x, t) = \sum_{\ell=0}^{\infty} C_\ell(x) t^{\ell+\gamma} \tag{20}$$

be one of the formal solutions of the modified hypergeometric system $H_{A,w}(\beta)$ constructed in Theorem 5 and $r + 1$ be the Gevrey index of $\psi(x, t)$ along T. We also assume $r\gamma \notin \mathbb{Z}$. Then the formal solution $\psi(x, t)$ is $1/r$-summable (as a formal power series in t) in all directions but finitely many directions for each $x \in U$ where U is an open set in the x-space $\mathbb{C}^n$. Furthermore its Borel sum determines a solution of the modified hypergeometric system $H_{A,w}(\beta)$.

Remark 12 *Under the assumption of Theorem (7),*

$$r = -\frac{|b_i|}{w \cdot b_i} \tag{21}$$

holds for any $i \notin \sigma$, where $\{b_i : i \notin \sigma\}$ is a basis of $\operatorname{Ker} A$ given in Remark 10. Therefore if $u \in \operatorname{Ker} A \cap \mathbb{Z}^n$, then $rw \cdot u$ is an integer.

Remark 13 *The condition that A_w contains $(1, 1, \ldots, 1)$ is assumed so that the Borel transformed series satisfies a regular holonomic system [9]. Hence, the growth condition (ii) of the Borel summability is satisfied because solutions of regular holonomic systems satisfy a polynomial growth condition. If we do not assume the condition, things become more complicated. See also Section 6.*

Remark 14 *The open set U can be chosen as follows. There exist constants c_{ij}, c_i, p_i, m such that the series φ_B, which will be defined in the proof below, converges when $(x_1, \ldots, x_n)$ and ζ satisfies $\sum_j c_{ij} \log|x_j| + c_i \log|\zeta| < p_i$, for $i = 1, \ldots, m$. Such constants exist because φ_B is a hypergeometric series which satisfies a regular holonomic A-hypergeometric system ([8], [16, Chapter 2]); see also forthcoming Lemma 7. Since only the non-negative powers of ζ modulo an exponent appear in φ_B, we may assume that $c_i > 0$. Therefore, there exists a cone C in $\mathbb{R}^n$ and a point p' such that the series φ_B converges when $(\log|x_1|, \ldots, \log|x_n|, \log|\zeta|) \in D := (p' + C) \times \{y \mid y < 1\}$. We may choose a domain $U \subset \mathbb{C}^n$ with compact closure and such that the set $\log|U| := \{(\log|x_i|) \mid x \in U\}$ is contained in $p' + C$.*

Let us prove Theorem 7. To study the analytic properties of $\hat{\mathcal{B}}_{1/r}[\psi]$, it is convenient to use

$$\varphi(x,z) := \psi(x,t)\Big|_{t=z^r} = \sum_{\ell=0}^{\infty} C_\ell(x) z^{r(\ell+\gamma)}. \tag{22}$$

Since $\psi(x,t)$ is a formal power series in $t^{w \cdot b_i}$, $z^{-r\gamma}\varphi(x,z)$ does not contain any fractional powers in z (cf. Remark 12). The Borel transform of φ with index 1 becomes

$$\hat{\mathcal{B}}_1[\varphi](x,\zeta) = \sum_{\ell=0}^{\infty} \frac{C_\ell(x)}{\Gamma(1 + r(\ell+\gamma))} \zeta^{r(\ell+\gamma)} = \hat{\mathcal{B}}_{1/r}[\psi](x, \zeta^r). \tag{23}$$

In the following, for the notational simplicity, we write $\varphi_B(x,\zeta)$ (resp., $\psi_B(x,\tau)$) instead of $\hat{\mathcal{B}}_1[\varphi](x,\zeta)$ (resp., $\hat{\mathcal{B}}_{1/r}[\psi](x,\tau)$).

Lemma 6 *For the power series φ given in* (22)*, we have*

$$\theta_\zeta \varphi_B = \hat{\mathcal{B}}_1[\theta_z \varphi].$$

If $r\gamma \neq 0, -1, -2, \ldots$, we also have

$$\frac{\partial \varphi_B}{\partial \zeta} = \hat{\mathcal{B}}_1[z^{-1}\varphi].$$

Proof. The first relation follows from

$$\begin{aligned}
\zeta \frac{\partial \varphi_B}{\partial \zeta}(x,\zeta) &= \sum_{\ell=0}^{\infty} \frac{C_\ell(x)}{\Gamma(1 + r(\ell+\gamma))} r(\ell+\gamma) \zeta^{r(\ell+\gamma)} \\
&= \mathcal{B}_1\left[\sum_{\ell=0}^{\infty} C_\ell(x) r(\ell+\gamma) z^{r(\ell+\gamma)}\right] = \mathcal{B}_1\left[z \frac{\partial \varphi}{\partial z}\right].
\end{aligned}$$

We also have

$$\begin{aligned}
\frac{\partial \varphi_B}{\partial \zeta}(x,\zeta) &= \sum_{\ell=0}^{\infty} \frac{C_\ell(x)}{\Gamma(1 + r(\ell+\gamma))} r(\ell+\gamma) \zeta^{r(\ell+\gamma)-1} \\
&= \sum_{\ell=0}^{\infty} \frac{C_\ell(x)}{\Gamma(r(\ell+\gamma))} \zeta^{r(\ell+\gamma)-1} \\
&= \hat{\mathcal{B}}_1\left[\sum_{\ell=0}^{\infty} C_\ell(x) z^{r(\ell+\gamma)-1}\right] = \hat{\mathcal{B}}_1\left[z^{-1}\varphi\right]. \qquad \textit{Q.E.D.}
\end{aligned}$$

Lemma 7 *The formal power series φ_B given in* (23) *formally satisfies the hypergeometric system $H_{A_B}(\beta_B)$, where*

$$A_B = \begin{pmatrix} A & 0 \\ w & -1/r \end{pmatrix}, \quad \beta_B = \begin{pmatrix} \beta \\ 0 \end{pmatrix}.$$

When the matrix A_B contains a rational number, we regard the $\mathbb{Z}$-module generated by the column vectors as the lattice to define the A-hypergeometric system. For example, when $A_B = \begin{pmatrix} 1 & 0 \\ 1 & 1/2 \end{pmatrix}$, $\beta_B = (\beta, 0)$ the lattice is $\mathbb{Z} \times \mathbb{Z}/2$ and the A-hypergeometric system is nothing but that for $A_B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\beta_B = (\beta, 0)$ for the lattice $\mathbb{Z}^2$.

Proof. It follows from Lemma 6 and relations $\theta_j \varphi = \theta_j \psi_{t=z^r}$, $\theta_z \varphi(x,z) = r(\theta_t \psi)|_{t=z^r}$ that

$$\begin{aligned}\left(\sum_{j=1}^n a_{ij}\theta_j - \beta\right)\varphi_B &= \hat{\mathcal{B}}_1\left[\left(\sum_{j=1}^n a_{ij}\theta_j - \beta\right)\varphi\right] \\ &= \hat{\mathcal{B}}_1\left[\left(\sum_{j=1}^n a_{ij}\theta_j - \beta\right)\psi\Big|_{t=z^r}\right] = 0\end{aligned}$$

and

$$\begin{aligned}\left(\sum_{i=1}^n w_i\theta_i - \frac{1}{r}\theta_\zeta\right)\varphi_B &= \hat{\mathcal{B}}_1\left[\left(\sum_{i=1}^n w_i\theta_i - \frac{1}{r}\theta_z\right)\varphi\right] \\ &= \hat{\mathcal{B}}_1\left[\left(\sum_{i=1}^n w_i\theta_i - \theta_t\right)\psi\Big|_{t=z^r}\right] = 0.\end{aligned}$$

Now we take vectors $u = (u_1, \ldots, u_{n+1})^T, v = (v_1, \ldots, v_{n+1})^T \in \mathbb{N}^{n+1}$ satisfying $A_B u = A_B v$. By its definition, we obtain

$$\frac{1}{r}(u_{n+1} - v_{n+1}) = \sum_{i=1}^n w_i(u_i - v_i) \in \mathbb{Z}.$$

Without loss of generality we may assume one of u_{n+1} and v_{n+1} is zero. Under this assumption, u_{n+1}/r and v_{n+1}/r are non-negative integers. Furthermore

$$\begin{aligned} & w_1u_1 + \cdots + w_nu_n - \frac{1}{r}u_{n+1} = w_1v_1 + \cdots + w_nv_n - \frac{1}{r}v_{n+1} \\ \iff \quad & w_1u_1 + \cdots + w_nu_n + \frac{1}{r}v_{n+1} = w_1v_1 + \cdots + w_nv_n + \frac{1}{r}u_{n+1}\end{aligned}$$

holds. Therefore

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \\ v_{n+1}/r \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ u_{n+1}/r \end{pmatrix} \in \mathbb{N}^{n+1} \quad \text{and} \quad \tilde{A}\begin{pmatrix} u_1 \\ \vdots \\ u_n \\ v_{n+1}/r \end{pmatrix} = \tilde{A}\begin{pmatrix} v_1 \\ \vdots \\ v_n \\ u_{n+1}/r \end{pmatrix}.$$

Hence

$$
\begin{aligned}
&\left(\partial_1^{u_1}\cdots\partial_n^{u_n}\partial_\zeta^{u_{n+1}}-\partial_1^{v_1}\cdots\partial_n^{v_n}\partial_\zeta^{v_{n+1}}\right)\varphi_B\\
&\quad=\hat{\mathcal{B}}_1\left[\left(\partial_1^{u_1}\cdots\partial_n^{u_n}z^{-u_{n+1}}-\partial_1^{v_1}\cdots\partial_n^{v_n}z^{-v_{n+1}}\right)\varphi\right]\\
&\quad=\hat{\mathcal{B}}_1\left[z^{-(u_{n+1}+v_{n+1})}\left(\partial_1^{u_1}\cdots\partial_n^{u_n}z^{v_{n+1}}-\partial_1^{v_1}\cdots\partial_n^{v_n}z^{u_{n+1}}\right)\varphi\right]\\
&\quad=\hat{\mathcal{B}}_1\left[z^{-(u_{n+1}+v_{n+1})}\left(\partial_1^{u_1}\cdots\partial_n^{u_n}t^{v_{n+1}/r}-\partial_1^{v_1}\cdots\partial_n^{v_n}t^{u_{n+1}/r}\right)\psi\Big|_{t=z^r}\right]\\
&\quad=0.
\end{aligned}
$$

Here we have used the second relation of Lemma 6. This completes the proof. Q.E.D.

Under the assumption of Theorem 7, $H_{A_B}(\beta_B)$ is regular holonomic. Therefore $\varphi_B(x,\zeta)$ is nothing but a GKZ series solution of $H_{A_B}(\beta_B)$, and $\varphi_B(x,\zeta)$ converges near $\zeta=0$ if $x\in U$ It also follows that the restriction of $\varphi_B(x,\zeta)$ to $\{x=x^0\}$, which is a function of ζ, satisfies some linear ordinary differential equation $E(x^0)$ of Fuchsian type. Let $\mathrm{Sing}\,(x^0)$ be all of the singular points of $E(x)$ except the origin and infinity, $\Theta(x)=\{\arg u;\ u\in\mathrm{Sing}\,(x)\}$. For any $\theta\in\mathbb{R}$ with $\theta\notin\Theta(x)$,

(i) $\varphi_B(x,\zeta)$ can be analytically continued to a sector $S(\theta,\delta)$ with some small $\delta>0$ since there is no singular point on $\{\zeta;\ \arg\zeta=\theta\}$.

(ii) The Borel transform $\varphi_B(x,\zeta)$ is of polynomial growth in $S(\theta,\delta)$ since a singular point of $E(x)$ is a regular singular point.

Hence we conclude that φ is Borel summable (i.e., 1-summable) in a direction θ. Since $\Theta(x)$ is a finite set for each fixed $x\in U$, φ is Borel summable in all directions except finite directions. Because of the relation (23), ψ is $1/r$-summable iff φ is 1-summable. Therefore ψ is $1/r$-summable in all directions except finite directions.

In the last we show that the Borel sum of ψ is a solution of the modified hypergeometric system $H_{A,w}(\beta)$ to finish the proof of Theorem 7. Because of the relation

$$
\mathcal{S}[\psi]\Big|_{t=z^r}=\mathcal{L}^\theta_{1/r}\hat{\mathcal{B}}_{1/r}[f]\Big|_{t=z^r}=\mathcal{L}^{\theta/r}_1\hat{\mathcal{B}}_1[\varphi]=\mathcal{S}[\varphi],
$$

Lemma 6 and Lemma 7, it is enough to prove (we omit the direction θ/r in the following.)

Lemma 8

$$
z\frac{\partial}{\partial z}\mathcal{L}_1[\varphi_B]=\mathcal{L}_1\left[\zeta\frac{\partial\varphi_B}{\partial\zeta}\right],\quad z^{-1}\mathcal{L}_1[\varphi_B]=\mathcal{L}_1\left[\frac{\partial\varphi_B}{\partial z}\right].
$$

Proof. To begin with, we give a proof in the case when $\Re(r\gamma)>0$. By differentiating under the integral sign we obtain

$$
\begin{aligned}
z\frac{\partial}{\partial z}\mathcal{L}_1[\varphi_B]&=\int_0^\infty\frac{\zeta}{z}e^{-\zeta/z}\varphi_B(x,\zeta)\frac{d\zeta}{z}-\int_0^\infty e^{-\zeta/z}\varphi_B(x,\zeta)\frac{d\zeta}{z}\\
&=-\int_0^\infty\frac{\partial}{\partial\zeta}(\zeta e^{-\zeta/z})\cdot\varphi_B(x,\zeta)\frac{d\zeta}{z}.
\end{aligned}
$$

By integral by parts, this equals to

$$-\left[\zeta e^{-\zeta/z}\cdot\frac{\varphi_B(x,\zeta)}{z}\right]_{\zeta=0}^{\infty}+\int_0^\infty e^{-\zeta/z}\cdot\zeta\frac{\partial\varphi_B}{\partial\zeta}(x,\zeta)\frac{d\zeta}{z}.$$

Since the boundary terms vanish (the boundary term coming from infinity vanishes because of the growth estimate of φ_B), the first relation follows. In a similar manner, we obtain

$$\begin{aligned}\mathcal{L}_1\left[\frac{\partial\varphi_B}{\partial z}\right] &= \int_0^\infty e^{-\zeta/z}\frac{\partial\varphi_B}{\partial\zeta}(x,\zeta)\frac{d\zeta}{z}\\ &= \left[e^{-\zeta/z}\frac{\varphi_B(x,\zeta)}{z}\right]_{\zeta=0}^{\infty}+\frac{1}{z}\int_0^\infty e^{-\zeta/z}\varphi_B(x,\zeta)\frac{d\zeta}{z}.\end{aligned}$$

The growth estimate of φ_B at infinity and the behavior of φ_B near the origin guarantee that the boundary terms vanish. This proves the second relation.

When $\Re(r\gamma)<0$, we use (19) as the definition of the Borel sum, and the same argument works. In this case all of the boundary terms come from infinity and they vanish. Q.E.D.

In the proof, we show that ψ_B is of polynomial growth. Therefore, for an arbitrary small positive c_2 we can find $c_1>0$ for which (13) holds with $f=\psi$, $\kappa=1/r$. Therefore, the condition (16) guarantees the following Corollary.

Corollary 6 *The Laplace integral* (14) *of the Borel sum of* ψ *converges in* $S(\theta,r\pi)$ *if* $\theta\notin\Theta(x)$. *In particular, we can set* $t=1$ *in the expression* $\mathcal{S}[\psi](t)$ *of the Borel sum of* ψ *if* $\Theta(x)$ *does not contain* 0. *The series* $\mathcal{S}[\psi](t)_{|t=1}$ *gives a solution of* $H_A(\beta)$ *and the formal series* $\psi(x,1)$ *also express the asymptotic expansion of the solution* $\mathcal{S}[\psi](t)_{|t=1}$ *along a curve* $(x_1,\dots,x_n)=(t^{w_1}c_1,t^{w_2}c_2,\dots,t^{w_n}c_n)$ *as* $t\to0$.

Remark 15 *Since the singular points* Sing (x) *of* $E(x)$ *depend on* x, *one of them may meets the path of integration of the Borel sum if* x *moves. In that case we obtain the analytic continuation of the Borel sum by deforming the path of integration. This is closely related to the Stokes phenomenon.*

Example 4 Put $A=(1,2)$. Then, the image of $\bar{A}^T$ is $\mathbb{R}^2$. Let us take $w=(0,1)$. A formal solution of the modified system $H_{A,w}(\beta)$ is

$$\psi(x,t)=x_1^\beta\sum_{m=0}^\infty\frac{[\beta]_{2m}}{m!}\left(\frac{x_2}{x_1^2}\right)^m t^m.$$

Since the Gevrey order $r+1$ of the modified system $H_{A,w}(\beta)$ along $t=0$ is 2, we set $z=t$ and $\varphi(x,z)=\psi(x,z)$. The Borel transform φ_B is

$$\begin{aligned}& x_1^\beta\sum_{m=0}^\infty\frac{[\beta]_{2m}}{m!}\left(\frac{x_2}{x_1^2}\right)^m\frac{\zeta^m}{m!}\\ =\;& x_1^\beta\cdot{}_2F_1(-\beta/2,(-\beta+1)/2,1;4x_2\zeta/x_1^2)\end{aligned}$$

The domain U may be defined by $\{(x_1, x_2) \mid -2\log|x_1| + \log|x_2| < -1, |x_1| < 1\}$. The series φ_B satisfies the A-hypergeometric system with $A_B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ and $\beta_B = (\beta, 0)^T$. The equation $E(x) = E(x_1, x_2)$ in the proof is

$$\left[(4x_2\zeta^2 - {x_1}^2\zeta)\left(\frac{\partial}{\partial\zeta}\right)^2 + \left((-4\beta+6)x_2\zeta - {x_1}^2\right)\frac{\partial}{\partial\zeta} + (\beta^2-\beta)x_2\right]\varphi_B(x,\zeta) = 0,$$

and

$$\mathrm{Sing}\,(x) = \left\{{x_1}^2/(4x_2)\right\}, \quad \Theta(x) = \{2\arg x_1 - \arg x_2\}.$$

The Borel sum of $\psi(x,t)$ is

$$\frac{x_1}{t}\int_0^{e^{i\theta}\cdot\infty} e^{-\tau/t}{}_2F_1(-\beta/2, (-\beta+1)/2, 1; 4x_2\tau/x_1^2)d\tau$$

and, for each $x \in U$, φ (and hence ψ) is Borel summable in all direction exept the angle $\theta = 2\arg x_1 - \arg x_2$.

The series $\psi(x,1)$ can be regarded as an asymptotic expansion of a solution of the original A-hypergeometric system for $A = (1,2)$ and β from (17) and Corollary 6. In other words, we have

$$x_1^\beta\int_0^{e^{i\theta}\infty} e^{-\tau}{}_2F_1\left(\frac{-\beta}{2}, \frac{-\beta+1}{2}, 1; \frac{4x_2\tau}{x_1^2}\right)d\tau \sim \psi(x,1),$$

which is a well-known asymptotic expansion.

Example 5 Put $A = (1,3,5,6)$. Then, the image of $\bar{A}^T$ is

$$\{(w_1, w_2, w_3, w_4) \mid w_2 - 3w_3 + 2w_4 = 0, w_1 - 5w_3 + 4w_4 = 0\}.$$

Let us take $w = (-4,-2,0,1)$ in the $\mathrm{Im}\,\bar{A}^T$. Then, the initial ideal $\mathrm{in}_{(-w,w)}(H_A(\beta))$ is generated by $\{\partial_{x_2}, \partial_{x_3}, \partial_{x_4}\}$ and $\theta_1 + 3\theta_2 + 5\theta_3 + 6\theta_4 - \beta$. The rank of this system is 1 and the solution of this system is spanned by x_1^β. We extend this solution to a series solution of $H_A(\beta)$. The solution can be written as

$$\sum_{u\in L}\frac{[\rho]_{u_-}}{[\rho+u]_{u_+}}x^{\rho+u} \tag{24}$$

$$\begin{aligned} = \; & x_1^\beta\Big(1 + \frac{\beta(\beta-1)(\beta-2)}{1!}x_1^{-3}x_2 \\ & + \frac{\beta(\beta-1)\cdots(\beta-5)}{2!}x_1^{-6}x_2^2 + \frac{\beta(\beta-1)\cdots(\beta-4)}{1!}x_1^{-5}x_3^1 + \cdots\Big) \end{aligned} \tag{25}$$

where $\rho = (\beta,0,0,0)$ and L is the length 4 vectors in

$$[0,[0,0,0,0]], [10,[-3,1,0,0]], [20,[-6,2,0,0]], [20,[-5,0,1,0]],$$
$$[25,[-6,0,0,1]], [30,[-9,3,0,0]], [30,[-8,1,1,0]], \ldots$$

Note that $p = w\cdot\ell$ in the entries of the form $[p, [\ell_1, \ell_2, \ell_3, \ell_4]]$.

The corresponding series solution of the modified system is

$$x_1^\beta t^{-4\beta}\big(1+\frac{\beta(\beta-1)(\beta-2)}{1!}x_1^{-3}x_2t^{10} \tag{26}$$
$$+\left(\frac{\beta(\beta-1)\cdots(\beta-5)}{2!}x_1^{-6}x_2^2+\frac{\beta(\beta-1)\cdots(\beta-4)}{1!}x_1^{-5}x_3^1\right)t^{20}+\cdots\big).$$

The Gevrey order $r = 1/\kappa$ of the modified system is $1/5$. Apply the Borel transformation (23). The transformed series is

$$x_1^\beta\zeta^{-4\beta/5}\big(\frac{1}{\Gamma(1-\frac{4}{5}\beta)}+\frac{\beta(\beta-1)(\beta-2)}{1!}x_1^{-3}x_2\frac{\zeta^2}{\Gamma(1-\frac{4}{5}\beta+2)} \tag{27}$$
$$+\quad\left(\frac{\beta(\beta-1)\cdots(\beta-5)}{2!}x_1^{-6}x_2^2+\frac{\beta(\beta-1)\cdots(\beta-4)}{1!}x_1^{-5}x_3^1\right)\frac{\zeta^4}{\Gamma(1-\frac{4}{5}\beta+4)}+\cdots\big).$$

The transformed function satisfies the A-hypergeometric system associated to the matrix

$$A_B=\begin{pmatrix} 1 & 3 & 5 & 6 & 0\\ -4 & -2 & 0 & 1 & -5\end{pmatrix} \tag{28}$$

and $\beta_B = (\beta, 0)$. It follows from the condition on the weight vector w that this $H_{A_B}(\beta_B)$ is a regular holonomic system. The series (27) can be obtained by taking the $(-u, u)$-initial ideal of $H_{A_B}(\beta_B)$ with respect to the weight vector $(w, 0)$ and $(0, 0, 1, 2, 0)$ as the tie breaking weight vector [16, Chapters 2,3].

The kernel element ℓ of A_B as a map from $\mathbb{R}^5$ to $\mathbb{R}^2$ is parametrized as

$$\ell_1=\ell_3+\frac{3}{2}\ell_4-\frac{3}{2}\ell_5,\ell_2=-2\ell_3-\frac{5}{2}\ell_4+\frac{1}{2}\ell_5.$$

Since we sum the series on $\ell_i \geq 0$, $i = 2, 3, 4, 5$ and $\ell \in \mathbb{Z}^5$, these lattice elements ℓ can be parametrized as

$$\ell_5=2m+e_p,\ell_4=2n+e_p,\ell_3=k,\ell_2=-2k-5n+m-2e_p,\ell_1=k+3n-3m$$
$$m,n\in\mathbb{N}\text{ and }2k+5n+2e_p\leq m$$

where $e_p = 0$ or 1. Let us introduce the following hypergeometric series

$$\sum_{m,n,k\geq 0,2k+5n+2e_p\leq m}\frac{z_3^k z_4^{2n+e_p}z_5^{2m+e_p}}{(b)_{k+3n-3m}(-2k-5n+m-2e_p)!k!(2n+e_p)!(a)_{2m+e_p}}$$

depending on an integer e_p. We denote it by $F_{\text{odd}}(a, b; z_3, z_4, z_5)$ when $e_p = 1$ and by $F_{\text{even}}(a, b; z_3, z_4, z_5)$ when $e_p = 0$. Then, the series (27) is expressed as

$$\frac{x_1^\beta\zeta^{-4\beta/5}}{\Gamma(1-\frac{4}{5}\beta)}F(x_1x_2^{-2}x_3,x_1^{3/2}x_2^{-5/2}x_4,x_1^{-3/2}x_2^{1/2}\zeta)$$

where

$$F(z)=F_{\text{odd}}(1-\frac{4}{5}\beta,\beta+1;z)+F_{\text{even}}(1-\frac{4}{5}\beta,\beta+1;z).$$

It follows from (17) that the series (26) is an asymptotic expansion of

$$\int_0^{e^{i\theta}\infty}e^{-(\tau/t)^5}\frac{x_1^\beta\tau^{-4\beta/5}}{\Gamma(1-\frac{4}{5}\beta)}F(x_1x_2^{-2}x_3,x_1^{3/2}x_2^{-5/2}x_4,x_1^{-3/2}x_2^{1/2}\tau)\frac{5\tau^4}{t^5}d\tau.$$

6 Borel transformation revisited

In this section we assume that the row span of the matrix A does not contain the vector $(1, 1, \dots, 1)$ and we will see that the study of the irregularity of the modified A-hypergeometric system along T allows us to give an analytic meaning to the Gevrey series solutions of the A-hypergeometric system $\mathcal{M}_A(\beta)$ along coordinate varieties constructed in [7]. More precisely, we have, see Remark 17, that they are asymptotic expansions of certain holomorphic solutions of $\mathcal{M}_A(\beta)$.

Let us start with an observation about the assumption of Theorem 7 that the row span of the matrix A does not contain the vector $(1, 1, \dots, 1)$, but that of A_w contains $(1, 1, \dots, 1)$. Since adding to w a linear combination of the rows of A does not change the ideal $H_{A,w,\alpha}(\beta)$ if we accordingly change α, we can assume without additional loss of generality that $w = \lambda(1, \dots, 1)$. On the other hand, for $\lambda \geq 0$ the modified system $\mathcal{M}_{A,w,\alpha}(\beta)$ is regular along T by Proposition 2 and Remark 2. Thus, we may further assume that $w = (-\kappa, \dots - \kappa)$ for some $\kappa \in \mathbb{Z}_{>0}$. For this w each of the formal series constructed in the proof of Theorem 5 is a Gevrey series along T with order $s = r + 1 = 1 + 1/\kappa$ multiplied by a term t^{γ}, $\gamma \in \mathbb{C}$. This sort of formal series solutions along T of the modified system $\mathcal{M}_{A,w,\alpha}(\beta)$ includes series of the form $t^{-\alpha}\phi(t^{w_1}x_1, \dots, t^{w_n}x_n)$ where $\phi(x)$ is a Gevrey solution of $M_A(\beta)$ along a coordinate variety of low dimension which is not Gevrey along a coordinate variety of greater dimension. These low dimensional coordinate varieties are described in Proposition 3.

Recall that a vector v is associated with σ if $v = v^{\mathbf{k}}$ is such that $v_i = k_i \in \mathbb{N}$ for all $i \notin \sigma$ and $Av = \beta$. Let us also recall that Δ_A is the convex hull of the columns of A and the origin while $\operatorname{conv}(A)$ is the convex hull of the columns of A. We have the following.

Proposition 3 *Take $w = (-\kappa, \dots, -\kappa)$ with $\kappa > 0$ and assume that β is very generic. Let σ be a d-simplex of A and v a vector associated with σ. Then, up to multiplication by a term t^{γ}, $\gamma \in \mathbb{C}$, the series $t^{-\alpha}\phi_v(t^{w_1}x_1, \dots, t^{w_n}x_n)$ is a formal series along T (in such a case, it is a Gevrey solution of $\mathcal{M}_{A,w,\alpha}(\beta)$ with index $s' = r' + 1 = 1 + 1/\kappa$) if and only if σ is contained in a facet of* $\operatorname{conv}(A)$ *that is not a facet of Δ_A.*

Proof. From [7], ϕ_v is a Gevrey solution of $\mathcal{M}_A(\beta)$ along $Y = \{x_i = 0 : |A_\sigma^{-1}a_i| > 1\}$ with index $s = r + 1 = \max_i\{|A_\sigma^{-1}a_i|\}$. In fact, for any simplex σ of A we can construct $\operatorname{vol}(\sigma)$ linearly independent Gevrey solutions as before.

When β is very generic these series have Gevrey index $s = r + 1 = \max_i\{|A_\sigma^{-1}a_i|\}$ and they are of the form ϕ_v for v associated with σ.

The series $t^{-\alpha}\phi_v(t^{w_1}x_1, \dots, t^{w_n}x_n)$ is annihilated by $H_{A,w,\alpha}(\beta)$ because ϕ_v is annihilated by $H_A(\beta)$. However, monomials appearing in ϕ_v are of the form x^{v+u} with u integer vectors in an affine translate of the positive span of the columns of the matrix B_σ. Thus the series $t^{-\alpha}\phi_v(t^{w_1}x_1, \dots, t^{w_n}x_n)$ is formal along T (up to multiplication by a term t^{γ}, $\gamma \in \mathbb{C}$) if and only if all the exponents of t belong to $\gamma + \mathbb{N}$. This happens if and only if $wu \in \mathbb{N}$ for all the columns u of B_σ. For $w = (-\kappa, \dots - \kappa)$ with $\kappa \in \mathbb{Z}_{>0}$ the scalar product of w and a column of B_σ is given by $wu = \kappa(|A_\sigma^{-1}a_i| - 1)$ for $i \notin \sigma$ and this product will be nonnegative for all $i \notin \sigma$ if and only if $|A_\sigma^{-1}a_i| \geq 1$ for all $i \notin \sigma$. This is equivalent to say that σ is a simplex of A such that all the columns of A are either in the hyperplane H_σ passing through the columns of A indexed by σ or in the corresponding single open half space not containing the origin. Equivalently, σ is contained in a facet of $\operatorname{conv}(A)$ that is not a facet of Δ_A. Q.E.D.

Let us consider a special type of vectors $w \in \mathbb{Z}^n$ that permits to relate the Gevrey solutions of the original A-hypergeometric system $\mathcal{M}_A(\beta)$ along coordinate varieties of arbitrary codimension to the Gevrey solutions along T of the corresponding modified system.

Given a simplex σ of A let us take a vector $w \in \mathbb{Z}^n$ with the following coordinates:

$$w_i = \begin{cases} |\det(A_\sigma)|(|A_\sigma^{-1}a_i| - 1) & \text{if } |A_\sigma^{-1}a_i| > 1 \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

up to addition with a linear combination of the rows of A. Notice that when σ is contained in a facet of $\operatorname{conv}(A)$ that is not a facet of Δ_A this vector w verifies the assumptions of Theorem 7. More precisely, $|\det(A_\sigma)|(-1,\ldots,-1) - w$ is a linear combination of the rows of A.

Remark 16 *Notice that for w given by (29) all the columns $\widetilde{a}_i$ of the matrix $\widetilde{A}(w)$ but $\widetilde{a}_{n+1}$ are contained in the union of at most two hyperplanes. More precisaly, if we take coordinates (y, y_{d+1}) in $\mathbb{R}^{d+1}$, where $y \in \mathbb{R}^d$ and $y_{d+1} \in \mathbb{R}$, we have that the hyperplane $\{y_{d+1} = 0\}$ contains all the columns $\widetilde{a}_i$ such that $|A_\sigma^{-1}a_i| \leq 1$ and the hyperplane $\{|A_\sigma^{-1}y| - \frac{1}{|\det(A_\sigma)|}y_{d+1} = 1\}$ contains $-|\det(A_\sigma)|\widetilde{a}_{n+1}$ and all the columns $\widetilde{a}_i$ such that $|A_\sigma^{-1}a_i| > 1$. In particular, the intersection of these two hyperplanes contains all the columns $\widetilde{a}_i$ such that $|A_\sigma^{-1}a_i| = 1$ (for example, all the columns $\widetilde{a}_i$ for $i \in \sigma$). We can also observe that the points $\{\widetilde{a}_i : i = 1,\ldots,n\} \cup \{-|\det(A_\sigma)|\widetilde{a}_{n+1}\}$ belong to the same hyperplane if and only if σ is contained in a facet of* $\operatorname{conv}(A)$ *that is not a facet of* Δ_A.

Lemma 9 *Let σ be any d-simplex of A and consider w given by (29). In this case, the unique slope of $\mathcal{M}_{A,w,\alpha}(\beta)$ along T is $s' = r' + 1 = 1 + 1/|\det(A_\sigma)|$.*

Proposition 4 *Let σ be any d-simplex of A and consider w given by (29) and β very generic. For any vector v associated with σ we have that, up to multiplication by a term t^γ, $\gamma \in \mathbb{C}$, the series $\psi(x,t) = t^{-\alpha}\phi_v(t^{w_1}x_1,\ldots,t^{w_n}x_n)$ is a Gevrey solution of $\mathcal{M}_{A,w,\alpha}(\beta)$ with index $s' = r' + 1 = 1 + 1/|\det(A_\sigma)|$ along T at any point of $T \cap U_{\sigma,R}$ for some $R > 0$, where $U_{\sigma,R} = \{(x,t) \in \mathbb{C}^n \times \mathbb{C} : \; |x_j t^{w_j}| < R|x_\sigma^{A_\sigma^{-1}a_j}|$ if $j \notin \sigma$ and $|A_\sigma^{-1}a_j| \geq 1\} \cap \{x_i \neq 0 : \; i \in \sigma\}$.*

Proof. By [7, Theorem 3.11] we have that ϕ_v is a Gevrey solution of $\mathcal{M}_A(\beta)$ with index $s = r + 1 = \max_i\{|A_\sigma^{-1}a_i|\})$ along $Y = \{x_i = 0 : \; |A_\sigma^{-1}a_i| > 1\}$ at any point of $Y \cap \{x \in \mathbb{C}^n : \; |x_j| < R|x_\sigma^{A_\sigma^{-1}a_j}|$ if $j \notin \sigma$ and $|A_\sigma^{-1}a_j| = 1\} \cap \{x_i \neq 0 : \; i \in \sigma\}$. Since β is very generic and v is associated with σ, we can assume without loss of generality that v is an exponent of $H_A(\beta)$ with respect to some generic perturbation $\widetilde{w}$ of w. Then $N_v = \mathbb{N}B_\sigma \cap \mathbb{Z}^n$. Recall that any column b_j of $B_\sigma = (b_j)_{j\notin\sigma}$ has coordinates $b_{i,j} = -(A_\sigma^{-1}a_j)_i$ for $i \in \sigma$, $b_{j,j} = 1$ and $b_{i,j} = 0$ for $i \notin \sigma \cup \{j\}$. It is clear from (29) that $wb_j = w_j \in \mathbb{N}$ for all $j \notin \sigma$. The conclusion follows then by using Remark 10. Q.E.D.

In analogy with Section 5 we denote $\psi_B(x,\tau) = \hat{\mathcal{B}}_{1/r'}[\psi](x,\tau)$, which defines a holomorphic function at any point in $U_{\sigma,R}$ for some $R > 0$ by Proposition 4. We also denote $\varphi(x,z) = \psi(x,z^{r'})$ and hence $\varphi_B(x,\zeta) = \hat{\mathcal{B}}_1[\varphi](x,\zeta) = \psi_B(x,\zeta^{r'})$ is convergent at points in the open set

$$U'_{\sigma,R} = \{(x,\zeta) \in \mathbb{C}^n \times \mathbb{C} : \; \zeta = \tau^{|\det(A_\sigma)|}, \; (x,\tau) \in U_{\sigma,R}\} \tag{30}$$

Moreover, the series $\varphi_B(x,z)$ is a holomorphic solution of the hypergeometric system $H_{A_B}(\beta_B)$ (in the variables (x,z)), where

$$A_B = \begin{pmatrix} A & 0 \\ w & -|\det(A_\sigma)| \end{pmatrix}, \quad \beta_B = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}.$$

Lemma 10 *For w given by (29) the hypergeometric system $H_{A_B}(\beta_B)$, is regular along T for all $\beta, \alpha \in \mathbb{C}$.*

In general the hypergeometric system $H_{A_B}(\beta_B)$ can have slopes along T'. However, we have the following.

Proposition 5 *$\varphi_B(x,z)$ has an analytic continuation which has polynomial growth close to T'.*

Proof. Let η be the set of (indices of) columns of A_B belonging to the hyperplane $\{|A_\sigma^{-1}y| - \frac{1}{|\det(A_\sigma)|}y_{d+1} = 1\}$ (see Remark 16) and denote this matrix by C. Let the cardinality of η be $q+1$. Notice that $n+1 \in \eta$. Recall that $\varphi_B(x,z)$ defines a holomorphic function at each point of $U'_{\sigma,R}$ (see Proposition 4 and (30)) and notice that $w_j = 0$ for all $j \notin \eta$

The hypergeometric system associated with C is regular holonomic for any parameter vector. We can write

$$\varphi_B(x,z) = \sum_{m \in \mathbb{N}^{n-q}} \varphi_m \frac{x_{\overline{\eta}}^m}{m!}$$

and $\varphi_m = \varphi_m(x_\eta)$ is a holomorphic solution of the hypergeometric system $H_C(\beta - \sum_{i \notin \eta} m_i a_i, \alpha)$. Then there exists a non-empty open set $W \subset \mathbb{C}^{q+1}$ such that, for all $m = (m_i)_{i \notin \eta} \in \mathbb{N}^{n-q}$, φ_m is holomorphic in W. We can choose W such that $W \times \mathbb{C}^{n-q} \subset U'_{\sigma,R}$ and then $\varphi_B(x,z)$ is convergent in $W \times \mathbb{C}^{n-q}$.

We also know that the singular locus of a hypergeometric system does not depend on the parameter but only on the matrix (see [1] and [8]) and hence we can consider an analytic continuation of $\varphi_B(x,z)$ close to $z = \infty$ which induces an analytic continuation of each φ_m close to $z = \infty$.

To simplify the exposition we will assume first that α is very generic. Each φ_m has polynomial growth since it is a solution of the regular hypergeometric system $M_C(\beta - \sum_{i \notin \eta} m_i a_i, \alpha)$. So, its convergent Nilsson series expansion at $z = \infty$ (see e.g. [16, Theorem 3.4.2]) can be written as a linear combination of series of the form $\phi_{v(m)}(x_\eta)$ (for some set of exponents $v(m)$ associated with a simplex of the matrix C) with support given by integer vectors in the kernel of C with coordinates sum equal to zero. Hence, we have a formal Nilsson series expansion $\widetilde{\varphi}_B(x,z)$ of $\varphi_B(x,1/z)$. We need to show that $\widetilde{\varphi}_B(x,z)$ is convergent in some open set, which implies that the growth of $\varphi_B(x,z)$ is polynomial close to $z = \infty$.

First of all, the support of $\widetilde{\varphi}_B$ is contained in a finite union of translates of the positive span of a basis of $\ker A_B$. Notice that, for a fixed $m \in \mathbb{N}^{n-q}$, the support of $\varphi_m(x_\eta)$ is given by the positive span of a basis of the kernel of C and thus $|u| = 0$ for each vector u in this support. As $m = (m_i)_{i \notin \eta} \in \mathbb{N}^{n-q}$ varies, each $m_i \in \mathbb{N}$ is multiplying a vector u in the kernel of A_B with $\operatorname{supp}(u) \subseteq \{i\} \cup \eta$ and $u_i = 1$. This fact, along with the fact that the i-th column of C is in the same half space than the origin with respect to the hyperplane

$\{|A_\sigma^{-1}y| - \frac{1}{|\det(A_\sigma)|}y_{d+1} = 1\}$, implies that $|u| \geq 0$ for u in $\ker(A_B)$. We conclude that $\widetilde{\varphi_B}(x,z)$ is convergent and it is a Nilsson series expansion of φ_B in the variables $x_1, \ldots, x_n, 1/z$. Thus $\varphi_B(x,z)$ has polynomial growth close to $z = \infty$.

Now, we can reduce the general case (when α is not necessarily generic) to the previous one following the ideas of the proof of [16, Theorem 3.5.1]. Q.E.D.

Remark 17 *Using Proposition 5 the arguments in Section 5 work also for w as in (29) instead of the assumption on w in Theorem 7. In particular we obtain an analogous version of Corollary 6.*

Example 6 *Put $A = (1\ 2\ 3)$, $\beta \in \mathbb{C}$ and $w = (0,0,1)$. The vector $v = (0, \beta/2, 0)$ is an exponent of the A-hypergeometric system $H_A(\beta)$ with respect to a perturbation of w and so the series*

$$\psi(x,t) = \phi_v(x_1, x_2, tx_3) = \sum_{m_1, m_3 \geq 0, (m_1+3m_3) \in 2\mathbb{Z}} \frac{[\beta/2]_{(m_1+3m_3)/2}}{m_1! m_3!} x_1^{m_1} x_2^{(\beta - m_1 - 3m_3)/2} x_3^{m_3} t^{m_3}$$

is one of the series considered in the proof of Theorem 5 and it is a Gevrey solution of the modified system $M_{A,w}(\beta)$ along T with order $s = r + 1 = 3$. Notice that $s = r + 1 = 3/2$ is the Gevrey index of $\psi(x,t)$ along T if and only if $\beta \notin 2\mathbb{N}$ (otherwise $\psi(x,t)$ is a polynomial). Following Section 5 but with our vector w (which does not satisfy the assumptions in Section 5 but is of the form (29) for $\sigma = \{2\}$) we consider the Borel transform of ψ with index $\kappa = 1/r = 2$:

$$\psi_B(x,\tau) = \sum_{m_1, m_3 \geq 0, (m_1+3m_3) \in 2\mathbb{Z}} \frac{[\beta/2]_{(m_1+3m_3)/2}}{m_1! m_3! \Gamma(1 + m_3/2)} x_1^{m_1} x_2^{(\beta - m_1 - 3m_3)/2} x_3^{m_3} \tau^{m_3}$$

is holomorphic in $\{(x,z) \in \mathbb{C}^4 : |\frac{x_3 t}{x_2^{3/2}}| < c, x_1, x_2 \neq 0\}$ for $c > 0$ small enough and it has an analytic continuation with respect to t to certain sector $S(\theta, \delta)$ and with polynomial growth in t. This follows from the fact that if we take $\varphi(x,z) := \psi(x,t)\Big|_{t=z^{1/2}}$ then its Borel transform (with index 1)

$$\varphi_B(x,\zeta) = \sum_{m_1, m_3 \geq 0, (m_1+3m_3) \in 2\mathbb{Z}} \frac{[\beta/2]_{(m_1+3m_3)/2}}{m_1! m_3! \Gamma(1 + m_3/2)} x_1^{m_1} x_2^{(\beta - m_1 - 3m_3)/2} x_3^{m_3} \zeta^{m_3/2}$$

is a solution of the hypergeometric system associated with A_B and $(\beta, 0)$ defined on $\mathbb{C}^4$ with coordinates (x_1, x_2, x_3, ζ). Notice φ_B has fractional powers in z but defines a multivalued holomorphic function in $\{(x,z) \in \mathbb{C}^4 : |\frac{x_3^2 z}{x_2^3}| < c, z \neq 0\}$ for $c > 0$ small enough.

As a consequence, the Borel sum of ψ with index 2 given by $\mathcal{S}[\psi](t) = \mathcal{L}_\kappa^\theta \hat{\mathcal{B}}_\kappa[f](t) = \int_0^{e^{i\theta}\cdot\infty} e^{-(\tau/t)^\kappa} \psi_B(x,\tau) d(\tau/t)^\kappa$ is a holomorphic solution of $M_{A,w}(\widetilde{\beta})$ and $\mathcal{S}[\psi](1)$ is a holomorphic solution of $M_A(\beta)$ which has an asymptotic expansion $\psi(x,1)$ that is Gevrey of order $s = 3/2$ along $x_3 = 0$.

References

[1] A. Adolphson, *A-hypergeometric functions and rings generated by monomials.* Duke Mathematical Journal 73 (1994), n. 2, p. 269-290.

[2] A. Assi, F.J. Castro-Jiménez and J.-M. Granger, *How to calculate the slopes of a D-module.* Compositio Math., 104 (1996) 107-123.

[3] W. Balser, From divergent power series to analytic functions, Lecture Notes in Mathematics **1582**, Springer, 1994.

[4] J. Bernstein, *Modules over the ring of differential operators. Study of the fundamental solutions of equations with constant coefficients.* Funkcional. Anal. i Priložen. 5 (1971), no. 2, 1–16.

[5] A. Dickenstein, F. Martínez and L. Matusevich, *Nilsson solutions for irregular A-hypergeometric systems.* To appear in Revista Matemática Iberoamericana, arXiv:1007.4225v2 [math.AG].

[6] A. Esterov, K. Takeuchi, *Confluent A-hypergeometric functions and rapid decay homology cycles,* arxiv:1107.0402, 2011.

[7] M. C. Fernández Fernández, *Irregular hypergeometric D-modules*, Adv. Math. 224 (2010) 1735-1764.

[8] I. M. Gel'fand, A. V. Zelevinsky, M. M. Kapranov, Hypergeometric functions and toral manifolds. Functional Analysis and its Applications **23**, 1989, 94–106.

[9] R. Hotta, Equivariant D-modules, arxiv:math/9805021, 1998.

[10] G. Laumon, *Transformations canoniques et spécialisation pour les D-modules filtrés.*, Astérisque **130**, (1985), 56–129.

[11] Y. Laurent, *Polygône de Newton et b-fonctions pour les modules microdifferéntiels*, Ann. Sci. École Norm. Sup. (4) **20** (1987), 391–441.

[12] Y. Laurent and Z. Mebkhout, *Pentes algébriques et pentes analytiques d'un $\mathcal{D}$-module.* Annales Scientifiques de L'E.N.S. 4^e série, tome 32, n. 1 (1999) p.39-69.

[13] L. F. Matusevich, E. Miller and U. Walther, *Homological methods for hypergeometric families.* J. Amer. Math. Soc. 18 (2005), 4, p.919-941.

[14] Z. Mebkhout, *Le théorème de positivité de l'irrégularité pour les D -modules.* Progress in. Math., vol. 88, p. 83-132. Birkhäuser, Boston, 1990.

[15] K. Ohara and N. Takayama, *Holonomic rank of A-hypergeometric differential-difference equations.* J. Pure Appl. Algebra 213 (2009), no. 8, 1536-1544.

[16] M. Saito, B. Sturmfels, and N. Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, Springer–Verlag, Berlin, 2000.

[17] M. Schulze and U. Walther, *Irregularity of hypergeometric systems via slopes along coordinate subspaces*. Duke Mathematical Journal 142, **3** (2008), 465-509.

[18] M. Schulze and U. Walther, *Irregularity of hypergeometric systems via slopes along coordinate subspaces*. arXiv:math/0608668v3 [math.AG].

[19] B. Sturmfels, Gröbner Bases and Convex Polytopes. American Mathematical Society, Univ. Lectures Series, Band 8. Providence, Rhode Island, 1996.

[20] N. Takayama, Modified $\mathcal{A}$-Hypergeometric Systems, Kyushu Journal of Mathematics **63** (2009), 113–122.