Local Gröbner fan

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Abstract

This note summarizes results concerning local Gröbner fans. Any Gröbner fan of an ideal in a power series ring, or in an analytic or formal differential operator ring, is found to be a polyhedral fan. We also compare the notions of global and local Gröbner fans, and discuss applications of our results. *To cite this article: R. Bahloul, N. Takayama, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Éventail de Gröbner local. Cette note résume des résultats portant sur l'éventail de Gröbner local. Nous montrons que l'éventail de Gröbner associé à un idéal d'un anneau de séries ou de l'anneau des opérateurs différentiels formels ou analytiques est un éventail polyédral. Nous comparons également les notions d'éventail de Gröbner global et local et discutons des applications de nos résultats. *Pour citer cet article : R. Bahloul, N. Takayama, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Version française abrégée

Le but de cette note est d'exposer des résultats portant sur l'éventail de Gröbner local. Les démonstrations se trouvent dans [5].

Étant donné un idéal polynomial $I \subset \mathbf{k}[x] = \mathbf{k}[x_1, \ldots, x_n]$ ($\mathbf{k}$ étant un corps de caractéristique nulle), Mora et Robbiano [10] ont introduit l'éventail de Gröbner. Il s'agit, étant donné un système de poids $u \in \mathbb{R}^n$ sur les variables x_i de considérer l'ensemble $C_I[u]$ des poids u' pour lesquels l'idéal initial $\mathrm{in}_{u'}(I)$ égale $\mathrm{in}_u(I)$. L'ensemble des $C_I[u]$ est l'éventail de Gröbner de I, que l'on qualifie d'ouvert ici et qu'on

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note $\mathcal{E}(I)$. En effet, l'ensemble des adhérences des $C_I[u]$ noté $\bar{\mathcal{E}}(I)$ s'appelle l'éventail de Gröbner fermé. Quand I est homogène, Sturmfels [12] a montré que ce dernier est un éventail polyédral (cf. Def. 0.1).

Dans le cas d'un idéal de $\mathbf{k}[[x]]$ (ou $\mathbb{C}\{x\}$) on peut faire une construction analogue mais où les systèmes de poids u doivent être dans $\mathbb{R}^n_{\leq 0}$ pour que les choses aient un sens (en effet, le poids d'une série est définie comme le maximum des poids des monômes qui la composent). On peut alors montrer que l'éventail de Gröbner est un éventail polyédral. Nous montrons ce même résultat pour un idéal de l'anneau (homogénéisé) des opérateurs différentiels formels (ou analytiques). Dans ce cas, les poids portent sur les x_i et les dérivées partielles avec des conditions de compatibilité. Ce dernier résultat complète ceux de Assi et al. [3] (où ils montrent seulement que l'éventail de Gröbner ouvert est une ensemble fini de cônes polyédraux rationels convexes). Mentionnons en passant que Saito et al. [11] ont fait de même dans le cas algébrique en complétant les résultats de Assi et al. [2]. Ces résultats sont rappelés dans la section 1 de la version anglaise.

Étant donné un idéal I du localisé (en 0 par exemple) $\mathbf{k}[x]_0$ de $\mathbf{k}[x]$, on peut montrer que l'éventail de Gröbner coïncide avec celui de l'extension formelle $\mathbf{k}[[x]]I$ de I. Si I est polynomial, on définit alors l'éventail de Gröbner local de I comme celui associé à $\mathbf{k}[x]_0 I$. Il existe des liens entre l'éventail de Gröbner global et local. Par exemple, si I est homogène, ils sont égaux modulo la stratification par les gradués de $\mathbf{k}[[x]]$. Sinon, on peut donner un exemple pour lequel cela n'a pas lieu. Ceci est présenté dans la section 2 de la version anglaise.

Dans la dernière section, on explique l'intérêt des résultats ci-dessus et évoque des applications. Ainsi on propose un algorithme de calcul de l'éventail de Gröbner local en se basant sur le fait qu'il soit polyédral. En effet, un éventail polyédral est entièrement défini par ses cônes maximaux ce qui réduit sensiblement la phase d'énumération des cônes. On évoque aussi l'application aux polynômes de Bernstein-Sato suivant [4] et celle par N. Touda à la notion de variété tropicale locale, cf [13].

Introduction

The goal of this note is to summarize a number of new results on local Gröbner fans. A more detailed and technical consideration of this work may be found in [5]. This paper deals with the notion of a Gröbner fan associated with an ideal in the ring of (convergent or formal) power series; or an ideal in the ring of differential operators with coefficients in a power series ring.

Roughly speaking, the Gröbner fan is a collection of polyhedral cones in the space of weight vectors, and each cone corresponds to a graded (or initial) ideal. A little more precisely, each cone is the equivalence class of weight vectors and two weight vectors belong to the same class if the corresponding initial ideals coincide.

Basically there are two contributions:

(1) We state that given an ideal in a power series ring or in a ring of (analytic of formal) differential operators, the Gröbner fan is a polyhedral fan (see Def. 0.1). These results are improvements of the result by Assi, Castro, Granger [3], who proved that such an ideal is a finite collection of polyhedral cones without proving that it is a polyhedral complex. This polyhedral property is essential for an efficient construction of the fan, and also for some applications such as local tropical varieties.

(2) We compare the notions of local and global Gröbner fans. Moreover, given a polynomial ideal or a left ideal in Weyl algebra, we propose an algorithm for computing its local Gröbner fan; that is the Gröbner fan of its formal extension.

The notion of a Gröbner fan for a polynomial ideal was introduced by Mora and Robbiano (see [10]) and revisited by Sturmfels in [12]. Let us first give some definitions: Let I be an ideal in $\mathbf{k}[x] = \mathbf{k}[x_1, \ldots, x_n]$, where $\mathbf{k}$ denotes a field (and all fields considered here are of characteristic 0). For a non-zero polynomial

$f = \sum_\alpha c_\alpha x^\alpha \in \mathbf{k}[x]$ (with $\alpha \in \mathbb{N}^n$, $c_\alpha \in \mathbf{k}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$), the support is $\mathrm{supp}(f) = \{\alpha \in \mathbb{N}^n | c_\alpha \neq 0\}$. We denote a weight vector on the variables x_i by an element $u \in \mathbb{R}^n$, and the space of such elements u is denoted by $\mathcal{U}_{\mathrm{glob}}$. With $u \in \mathcal{U}_{\mathrm{glob}}$, we associate a weight order on the non-zero elements of $\mathbf{k}[x]$: $\mathrm{ord}^u(f) = \max\{u \cdot \alpha | \alpha \in \mathrm{supp}(f)\}$ (here $u \cdot \alpha = \sum_1^n u_i \alpha_i$). This gives rise to a filtration $F^u_{\leq \lambda}(\mathbf{k}[x]) = \{f | \mathrm{ord}^u(f) \leq \lambda\}$, where λ is a real number, and to a graded ring $\oplus_\lambda F^u_{\leq\lambda}(\mathbf{k}[x])/F^u_{<\lambda}(\mathbf{k}[x])$ (the notation $F^u_{<\lambda}$ is natural). Notice that $\mathrm{gr}^u(\mathbf{k}[x]) = \mathbf{k}[x]$.

Given an ideal I in $\mathbf{k}[x]$, there is an induced filtration and an induced graded (or initial) ideal $\mathrm{in}_u(I) \subset \mathrm{gr}^u(\mathbf{k}[x])$. (Notice that $\mathrm{in}_u(I)$ is generated by all the initial forms $\mathrm{in}_u(f)$, when f runs over I. Here $\mathrm{in}_u(f) = \sum_{u\cdot\alpha = \mathrm{ord}^u(f)} c_\alpha x^\alpha$.) This enables us to consider an equivalence relation on $\mathcal{U}_{\mathrm{glob}}$: $u \sim u'$ if and only if $\mathrm{in}_u(I) = \mathrm{in}_{u'}(I)$. Let us denote by $C_I[u]$ the equivalence class of some $u \in \mathcal{U}_{\mathrm{glob}}$. We call the set of closures (in Euclidean topology) of all distinct $C_I[u]$, $u \in I$, the closed Gröbner fan of I, and we shall denote it by $\bar{\mathcal{E}}(I, \mathcal{U}_{\mathrm{glob}})$ or simply $\bar{\mathcal{E}}(I)$. The Gröbner fan of I in $\mathbf{k}[x]$ is also called a *global* Gröbner fan to distinguish it from the concept of a local Gröbner fan, which will be defined later in this paper.

Let us give some comments here. It is known (see Sturmfels [12]) that if I is homogeneous (for the total degree) then $\bar{\mathcal{E}}(I)$ is a polyhedral fan.

Definition 0.1 *A polyhedral fan is a finite collection of (relatively closed) polyhedral cones satisfying:*
– *Any face of any cone is again a cone.*
– *The intersection of any two cones is a face of both.*

However, it is not true in general that $\bar{\mathcal{E}}(I)$ is a polyhedral fan. The following example is well known.

Example 1 Let I be the ideal of $\mathbf{k}[x_1, x_2]$ generated by $\{1 + x_1, 1 + x_2\}$. It is then easy to check that $\bar{\mathcal{E}}(I) = \{\{0\}, \mathbb{R}^2 \smallsetminus (\mathbb{R}_{>0}e_1 + \mathbb{R}_{>0}e_2), \mathbb{R}_{\geq 0}e_1, \mathbb{R}_{\geq 0}e_2, \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2\}$ where $e_1 = (1,0)$ and $e_2 = (0,1)$. In this list, the second "cone" is not convex.

By an abuse of notation, the set $\mathcal{E}(I) := \{C_I[u] | u \in \mathcal{U}_{\mathrm{glob}}\}$ is called the open Gröbner fan, even though it is not a polyhedral fan.

Chronologically, the Gröbner fan was first defined for polynomial ideals and then for ideals in Weyl algebra (the ring of differential operators with polynomial coefficients), see Assi, Castro and Granger [2]. It was revisited independently by Saito, Sturmfels and Takayama [11]. Meanwhile, the preprint [1] by Assi treated the case of formal power series. The analytic (and formal) differential case was subsequently treated by Assi et al. [3].

What is the contribution of our paper with respect to these works? In the remainder of this introduction we discuss what the present study adds to this subject area.

Mora and Robbiano [10] proved (for the polynomial case) that given a homogeneous ideal, the open Gröbner fan is made up of a finite number of open convex polyhedral rational cones, but they did not prove that the closed Gröbner fan is polyhedral. This last fact was proved by Sturmfels [12]. Concerning the differential case, the definitions are analogous, except that the set of weight vectors $\mathcal{W}_{\mathrm{glob}}$ (or $\mathcal{W}_{\mathrm{loc}}$) is a subset of $\mathbb{R}^{2n}$ due to the non-commutativeness of the ring. In order to have interesting and useful results we need homogeneity (as in the polynomial case), which is why we usually introduce an extra variable h and, for the case of Weyl algebra, assume the ideal to be homogeneous with respect to the whole set of variables x_i, ∂_i, h, where ∂_i is the partial differential operator with respect to x_i. For the analytic or formal case we assume the ideal to be homogeneous with respect to ∂_i, h.

For the global case (the homogenized Weyl algebra), Assi et al. [2] proved similar results to those of Mora and Robbiano [10] concerning the open Gröbner fan. Note that they called it a Gröbner fan. In [11], Saito et al. independently showed that the closed Gröbner fan is a polyhedral fan. In Assi et al. [3], again, the authors proved similar results for the open Gröbner fan in the analytic and formal differential situation. The same result was presented in the preprint by Assi [1] for power series rings.

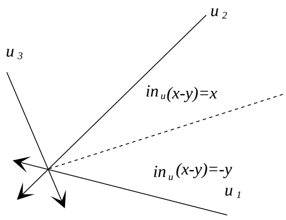
Note finally that in the recent preprint by Jensen, it is proved that Gröbner fans are generally not normal fans of convex polyhedra (see [8])

In this note, we will present the result that the closed Gröbner fan is a polyhedral fan in the analytic and formal commutative and differential case.

1. Local Gröbner fan is a polyhedral fan

Here we describe the two main results of the first part of [5].

Figure 1.



Commutative case

For simplicity, we consider the formal version only. Let I be an ideal in $\mathbf{k}[[x]] = \mathbf{k}[[x_1, \dots, x_n]]$. Define $\mathcal{U}_{\mathrm{loc}} = \{u \in \mathbb{R}^n | u_i \leq 0, \forall i = 1, \dots, n\}$ as the space of local weight vectors. For a non-zero $f = \sum_\alpha c_\alpha x^\alpha \in \mathbf{k}[[x]]$, the support is $\mathrm{supp}(f) := \{\alpha \in \mathbb{N}^n | c_\alpha \neq 0\}$, the u-order is $\mathrm{ord}^u(f) := \max_{\mathrm{supp}(f)}\{u \cdot \alpha\}$, the initial form is $\mathrm{in}_u(f) := \sum_{u\cdot\alpha = \mathrm{ord}^u(f)} c_\alpha x^\alpha$.
As in the introduction, the u-order naturally defines a filtration F^u on $\mathbf{k}[[x]]$, and a graded ring. Notice that, in contrast to the global case, $\mathrm{gr}^u(\mathbf{k}[[x]])$ is not isomorphic to $\mathbf{k}[[x]]$.
In fact we can identify it with $\mathbf{k}[[x_{i_{m+1}}, \dots, x_{i_n}]][x_{i_1}, \dots, x_{i_m}]$ where: $u_i < 0 \iff i \in \{i_1, \dots, i_m\}$. The induced filtration on I gives rise to a graded ideal $\mathrm{in}_u(I) \subset \mathrm{gr}^u(\mathbf{k}[[x]])$. Notice that $\mathrm{in}_u(I)$ is generated by the set of $\mathrm{in}_u(f)$ when f runs over I.

The definition of the (open or closed) Gröbner fan is stated as follows. Two local weight vectors u and u' are said to be equivalent if $\mathrm{gr}^u(\mathbf{k}[[x]])$ identifies to $\mathrm{gr}^{u'}(\mathbf{k}[[x]])$ (in the above sense) and if $\mathrm{in}_u(I)$ equals $\mathrm{in}_u(I)$. This last condition makes sense since these ideals are in the "same" ring by the first condition.

If $C_I[u]$ denotes the equivalence class of some $u \in \mathcal{U}_{\mathrm{loc}}$, the local open (resp. closed) Gröbner fan denoted by $\mathcal{E}(I) = \mathcal{E}(I, \mathcal{U}_{\mathrm{loc}})$ (resp. $\bar{\mathcal{E}}(I) = \bar{\mathcal{E}}(I, \mathcal{U}_{\mathrm{loc}})$) is the set of all distinct $C_I[u]$ (resp. $\overline{C_I[u]}$) when u runs over $\mathcal{U}_{\mathrm{loc}}$.

Theorem 1.1 ([5, Th. 2.0.3]) *The set $\bar{\mathcal{E}}(I)$ is a rational polyhedral fan.*

Similar results occur for an ideal in $\mathbb{C}\{x\}$.

Example 2 The local Gröbner fan $\bar{\mathcal{E}}(\mathbf{k}[[x, y, z]]\langle x - y, 1 - z\rangle)$ is the collection of all faces of the negative orthant $Q = \{(u_1, u_2, u_3)|u_i \leq 0\}$, but $\bar{\mathcal{E}}(\mathbf{k}[x, y, z]\langle x - y, 1 - z\rangle) \cap Q$ is not a rational polyhedral fan. (see figure 1).

Analytic or formal differential case

Let $h_{(\mathbf{0},\mathbf{1})}(\hat{D})$ be the $\mathbf{k}[[x]]$-algebra generated by the ∂_i's and a new variable h with the following non-trivial relations: $[\partial_i, a(x)] = \frac{\partial a(x)}{\partial x_i} \cdot h$, $a(x) \in \mathbf{k}[[x]]$,$1 \leq i \leq n$. This is the $(\mathbf{0}, \mathbf{1})$-homogenized version of $\hat{D} = \mathbf{k}[[x]][\partial_1, \dots, \partial_n]$. Here $(\mathbf{0}, \mathbf{1})$ means that we look at the total degree with respect to the ∂_i (i.e. where the weight of the x_i's is 0 and that of the ∂_i's is 1). Therefore $h_{(\mathbf{0},\mathbf{1})}(\hat{D})$ is graded with respect to the weight vector $(\mathbf{0}, \mathbf{1}, 1)$. Thus, let J be a homogeneous (or graded) ideal of $h_{(\mathbf{0},\mathbf{1})}(\hat{D})$. Put $\mathcal{W}_{\mathrm{loc}} = \{w = (u, v) \in \mathbb{R}^{2n} \mid u_i \leq 0, u_i + v_i \geq 0\}$. Then $\mathcal{W}_{\mathrm{loc}}$ is the set of admissible weight vectors. To any $w \in \mathcal{W}_{\mathrm{loc}}$, we can associate a filtration on $h_{(\mathbf{0},\mathbf{1})}(\hat{D})$ which induces one on J, and consequently a graded ring $\mathrm{gr}^w(h_{(\mathbf{0},\mathbf{1})}(\hat{D}))$ and a graded (or initial) ideal $\mathrm{in}_w(J)$. As before, we associate an equivalence relation on $\mathcal{W}_{\mathrm{loc}}$ by the equality of the graded rings $gr^\bullet(h_{(\mathbf{0},\mathbf{1})}(\hat{D}))$ and the initial ideals $\mathrm{in}_\bullet(J)$. This gives rise to a partition of $\mathcal{W}_{\mathrm{loc}}$ called the open Gröbner fan of J and denoted by $\mathcal{E}(J) = \mathcal{E}(J, \mathcal{W}_{\mathrm{loc}})$. The set of closures of $C \in \mathcal{E}(J, \mathcal{W}_{\mathrm{loc}})$, denoted by $\bar{\mathcal{E}}(J) = \bar{\mathcal{E}}(J, \mathcal{W}_{\mathrm{loc}})$, is called the (local) closed Gröbner fan.

Theorem 1.2 ([5, Th. 3.0.2]) *The set $\bar{\mathcal{E}}(J, \mathcal{W}_{\mathrm{loc}})$ is a rational polyhedral fan.*

Note that Assi et al. [3] defined the "analytic standard fan" of a given ideal $I \subset \mathcal{D}_0$ (ring of analytic differential operators) as the open Gröbner fan of its homogenization $h_{(\mathbf{0},\mathbf{1})}(I)$ with respect to the total degree in the ∂_i's.

Let us end this paragraph with few words on the proof of this theorem (the proof of Th. 1.1 is similar). We adapted Sturmfels' [12] proof for the polynomial setting to our situation: the main steps are analogous, but techniques for proofs are different. We stated Janet type divisions in the bi-graded ring $\mathrm{gr}^w(h_{(\mathbf{0},\mathbf{1})}(\hat{D}))$, and we used the notion of standard bases.

2. General results on local and global Gröbner fans - Algorithm

Here we are concerned with the results of the second part of [5]. Let us first introduce some notation. We denote by $\mathbf{k}[x]_0$ the localization at 0 of $\mathbf{k}[x]$, and $D_{\{0\}}$ the subring of $\hat{D}$ made of differential operators with coefficients in $\mathbf{k}[x]_0$. This subring is the localization at 0 of D, the ring of polynomial differential operators. Moreover, for a given ideal I, e.g. in $\mathbf{k}[x]$, and a subset $U \subset \mathcal{U}_{\mathrm{glob}}$, $\mathcal{E}(I, U)$ is the restriction to U of $\mathcal{E}(I, \mathcal{U}_{\mathrm{glob}})$.

Let I be an ideal in $\mathbf{k}[x]$, we call $\mathcal{E}(\mathbf{k}[x]_0 I) = \mathcal{E}(\mathbf{k}[x]_0 I, \mathcal{U}_{\mathrm{loc}})$ the local (open) Gröbner fan of I. Similarly, if I is a (left) ideal in D, we call $\mathcal{E}(D_{\{0\}} I) = \mathcal{E}(D_{\{0\}} I, \mathcal{W}_{\mathrm{loc}})$ the local (open) Gröbner fan of I. In fact, we can also consider analogous constructions. Indeed, if I is generated by $(\mathbf{0},\mathbf{1})$-homogeneous elements of D, then we can consider $\mathcal{E}(h_{(\mathbf{0},\mathbf{1})}(D_{\{0\}})I, \mathcal{W}_{\mathrm{loc}})$ and, in general, we can consider $\mathcal{E}(h_{(\mathbf{0},\mathbf{1})}(D_{\{0\}}I), \mathcal{W}_{\mathrm{loc}})$.

Here we shall recall "commutative results", i.e. when I is an ideal in $\mathbf{k}[x]$, $\mathbf{k}[x]_0$ or $\hat{O}$.
The definition above is justified by the following (see [5, Prop. 4.1.8]):

- If $I \subset \mathbf{k}[x]_0$ then $\mathcal{E}(I, \mathcal{U}_{\mathrm{loc}}) = \mathcal{E}(\hat{O}I, \mathcal{U}_{\mathrm{loc}})$.

Let us denote by $\mathcal{U}'_{\mathrm{loc}}$ the interior of $\mathcal{U}_{\mathrm{loc}}$ then (see [5, Cor. 4.1.6]):

- $\mathcal{E}(I, \mathcal{U}'_{\mathrm{loc}}) = \mathcal{E}(\mathbf{k}[x]_0 I, \mathcal{U}'_{\mathrm{loc}})$, if $I \subset \mathbf{k}[x]$.

This means that global and local Gröbner fans coincide in $\mathcal{U}'_{\mathrm{loc}}$. So if there exists a difference between these two concepts, it should be on the border of $\mathcal{U}_{\mathrm{loc}}$. The following example is trivial but gives an idea of what may happen. Let us denote by $S_{\hat{O}} = \mathcal{E}(\hat{O}1, \mathcal{U}_{\mathrm{loc}})$ the stratification of $\mathcal{U}_{\mathrm{loc}}$ defined by the equality of the graded rings in $\hat{O}$.

- We have $\mathcal{E}(\mathbf{k}[x]_0 1) = \mathcal{E}((1), \mathcal{U}_{\mathrm{loc}}) \cap S_{\hat{O}}$ (see [5, Ex. 4.1.7]).

For a general polynomial ideal I, we do not have $\mathcal{E}(\mathbf{k}[x]_0 I) = \mathcal{E}(I, \mathcal{U}_{\mathrm{loc}}) \cap S_{\hat{O}}$ (see the same example in loc. cit.). However, when the ideal I is homogeneous this equality holds (see [5, Th. 5.1.1]).

Let us complete this exposition by saying that in [5] we give analogous results in the differential case (except for [5, Th. 5.1.1]), see sections 4 and 5 in loc. cit..

Algorithm

Here, we shall describe an algorithm for computing the local Gröbner fan of a polynomial ideal I. Let L be a linear subspace of $\mathbb{R}^n$. Note that since the local Gröbner fan $\bar{\mathcal{E}}(\mathbf{k}[x]_0 I)$ is a polyhedral fan, the restriction of it to L is also a polyhedral fan. In most applications, we need to compute the restriction of the local Gröbner fan $\bar{\mathcal{E}}(\mathbf{k}[x]_0 I) \cap L$. We will present our algorithm for this problem. We have also treated the differential cases in [5, subsec. 5.2 & 5.4].

Let $I = \mathbf{k}[x] \cdot \{f_1, \ldots, f_q\}$. We denote by $I^{(h)}$ the ideal of $\mathbf{k}[x, h]$ generated by the homogenization (for the total degree, or, in general, a weight-degree with positive weights) of the f_i's. We can prove ([5, Prop. 5.1.2]) that the global Gröbner fan $\mathcal{E}(I^{(h)})$ refines $\mathcal{E}(I)$ and (see [5, Rem. 4.1.10]) $\mathcal{E}(I, \mathcal{U}_{\mathrm{loc}})$ refines the local Gröbner fan $\mathcal{E}(\mathbf{k}[x]_0 I)$. Thus $\mathcal{E}(I^{(h)}, \mathcal{U}_{\mathrm{loc}})$ refines $\mathcal{E}(\mathbf{k}[x]_0 I)$ and we have the following algorithm (see [5, Algo. 5.1.4]):

– Step 1: Compute the set $\mathcal{E}_0$ of maximal cones of the global Gröbner fan $\bar{\mathcal{E}}(I^{(h)}, \mathcal{U}_{\mathrm{loc}} \cap L)$ (restricted to $\mathcal{U}_{\mathrm{loc}} \cap L$).

– Step 2: For each $C \in \mathcal{E}_0$, take any $u \in C$ and consider $I_C = \mathrm{in}_u(\mathbf{k}[x]_0 I)$. Glue together all the C in $\mathcal{E}_0$ that have the same I_C (this comparison can be done by an écart division). Thus, we obtain the maximal cones of $\bar{\mathcal{E}}(\mathbf{k}[x]_0 I) \cap L$ from which we can reconstruct the whole fan $\bar{\mathcal{E}}(\mathbf{k}[x]_0 I) \cap L$.

3. Interest and applications

The algorithm above is strongly dependent on the fact that the local Gröbner fan is polyhedral, because we work with maximal dimensional cones in $\mathcal{U}_{\mathrm{loc}} \cap L$.

There is another application of the result in section 1. N. Touda [13] explicitly used Th. 1.1 to study the notion of local tropical varieties.

Moreover, our algorithmic results in the differential case enable the computation of the local Bernstein-Sato polynomial following the results in [4]. Let us notice that the local Bernstein-Sato polynomial constructed in loc. cit. is a product of local b-functions associated with the 1-dimensional cones of the local Gröbner fan of some ideal restricted to a subspace, say V, that lies in the border of $\mathcal{W}_{\mathrm{loc}}$. In [5], we computed an example where the global Gröbner fan restricted to V is trivial, but the local Gröbner fan restricted to V is not. This result again shows that the border contains highly non-trivial information.

We also note that the Gröbner fan of analytic (or formal) differential ideals is strongly related to the notion of slopes in D-module theory. Indeed, the set of geometric slopes along a hypersurface are contained in the set of the 1-faces of the restriction of the Gröbner fan to a 2-dimensional subspace of $\mathcal{W}_{\mathrm{loc}}$ (see e.g. [6, §3.5] and the references in there).

To finish, let us mention that Jensen has developed a software package called Gfan [9] (see also Fukuda et al. [7]), which can compute the (global) Gröbner fan of a polynomial ideal. In [7], the authors propose a theory of Gröbner fans for non-homogeneous ideals, but these fans are not local.

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